

Tuesday, February 13, 2018

Noether's Theorem

[peskin & schroeder p 17-19]

Consider the continuous transformation of a field $\phi(x)$:

$$\phi(x) \rightarrow \phi'(x) = \phi(x) + \alpha \Delta\phi(x)$$

this transformation is called a continuous symmetry of the field / system, if it does not change the field equations of motion.

$$\text{i.e. } \mathcal{L}(\phi(x)) \rightarrow \mathcal{L}(\phi'(x)) = \mathcal{L}(\phi(x)) + \alpha \Delta\mathcal{L}'$$

where

$$\left\{ \begin{array}{l} \Delta\mathcal{L}' = 0 \\ \text{or} \\ \Delta\mathcal{L}' = \partial_\mu J^\mu \end{array} \right.$$

some general "current"



you can generally figure this out
by direct calculation.

However, we can calculate $\delta L'$ from differential calculus:

$$\begin{aligned} \delta L = dL' &= d \left(\frac{\partial L}{\partial \phi} \delta \phi + \frac{\partial L}{\partial (\partial_\mu \phi)} \underbrace{\partial_\mu(\delta \phi)}_{\Delta(\partial_\mu \phi)} \right) \\ \delta L' &= \frac{\partial L}{\partial \phi} \delta \phi + \frac{\partial L}{\partial (\partial_\mu \phi)} \underbrace{\partial_\mu(\delta \phi)}_{\Delta(\partial_\mu \phi)} + \partial_\mu \left(\frac{\partial L}{\partial (\partial_\mu \phi)} \right) \delta \phi - \partial_\mu \left(\frac{\partial L}{\partial (\partial_\mu \phi)} \right) \delta \phi \\ &= \frac{\partial L}{\partial \phi} \delta \phi + \partial_\mu \left[\left(\frac{\partial L}{\partial (\partial_\mu \phi)} \right) \delta \phi \right] - \partial_\mu \left(\frac{\partial L}{\partial (\partial_\mu \phi)} \right) \delta \phi \\ &= \underbrace{\partial_\mu \left[\left(\frac{\partial L}{\partial (\partial_\mu \phi)} \right) \delta \phi \right]}_{\text{identify this term with } \partial_\mu J^\mu} + \underbrace{\left[\frac{\partial L}{\partial \phi} - \partial_\mu \left(\frac{\partial L}{\partial (\partial_\mu \phi)} \right) \right] \delta \phi}_{=0} \end{aligned}$$

since it is the Euler-Lagrange equation for the field.

$$\begin{aligned} \text{Thus } \partial_\mu J^\mu &= \partial_\mu \left[\left(\frac{\partial L}{\partial (\partial_\mu \phi)} \right) \delta \phi \right] \quad \text{sometimes } J^\mu = 0 \\ \Rightarrow \partial_\mu \left[\underbrace{\frac{\partial L}{\partial (\partial_\mu \phi)} \delta \phi - J^\mu}_{J^\mu} \right] &= 0 \quad \text{determined by direct calculation} \\ \Leftrightarrow \partial_\mu J^\mu &= 0 \quad \text{local conservation law in the form of a continuity equation} \\ \Rightarrow Q_j &= \int \int^0 dx^3 = \text{constant in time} \\ &\quad = \text{conserved quantity / "charge"} \end{aligned}$$

$$\text{quick proof: } \partial_\mu J^\mu = 0 \Leftrightarrow \frac{\partial}{\partial x^\mu} J^\mu = 0$$

$$\Leftrightarrow \frac{1}{c} \frac{\partial}{\partial t} \int^0 J^0 + \frac{\partial}{\partial x^1} J^1 + \frac{\partial}{\partial x^2} J^2 + \frac{\partial}{\partial x^3} J^3 = 0$$

$$\Leftrightarrow \frac{1}{c} \frac{\partial}{\partial t} \int^0 J^0 + \vec{\nabla} \cdot \vec{J} = 0$$

$$\Rightarrow \underbrace{\frac{1}{c} \frac{\partial}{\partial t} \int_{\text{all space}}^0 J^0 d^3x}_{Q_f} + \underbrace{\int_{\text{all space}} (\vec{\nabla} \cdot \vec{J}) d^3x}_{\int \vec{J} \cdot d\vec{s}} = 0 \quad \text{divergence theorem}$$

$\int \vec{J} \cdot d\vec{s} = 0$ (i.e. nothing gets out of universe)

bounding surface (at infinity)

$$\Rightarrow \frac{\partial}{\partial t} Q_f = 0 \Rightarrow Q_f = \text{constant in time.}$$

$$\text{Example 1: } \mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2$$

Consider the transformation $\phi \xrightarrow{\phi'=\phi+\alpha} \phi + \alpha$ $\alpha = \text{cst}$

$$= \phi + \alpha \frac{\Delta \phi}{\equiv 1}$$

$$\mathcal{L}' = \frac{1}{2} (\partial_\mu (\phi + \alpha))^2 = \frac{1}{2} (\partial_\mu \phi)^2 = \mathcal{L} \underbrace{+ \partial_\mu (J^\mu = 0)}$$

\hookrightarrow Noether's theorem: $J^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \underset{\equiv 1}{\cancel{\partial_\mu \phi}}$

$\Rightarrow \partial_\mu \phi$ is a conserved current

$$\hookrightarrow \partial_\mu J^\mu = \partial_\mu (\partial_\mu \phi) = 0$$

$$\Rightarrow \int_{\text{all space}} \partial_\mu \phi d^3x = \text{constant in time}$$

$\Rightarrow \frac{\partial}{\partial t} \int_{\text{all space}} \phi d^3x = \text{constant} \Rightarrow$ conservation of flux/flow probability or particle number

$$\begin{aligned}\text{Example 2: } \mathcal{L} &= |\partial_\mu \phi|^2 - m^2 |\phi|^2 \\ &= (\partial_\mu \phi)(\partial_\mu \phi^*) - m^2 \phi \phi^*\end{aligned}$$

Consider the transformation: $\phi \rightarrow e^{i\alpha} \phi$, $\alpha = \text{const}$

for α small $\phi \rightarrow \phi' = \phi + i\alpha \phi$

$$\alpha \Delta \phi$$

$$\Delta \phi = i\alpha \phi$$

likewise:

$$\phi^* \rightarrow e^{-i\alpha} \phi^*$$

for α small $\phi^* \rightarrow \phi'^* = \phi^* - i\alpha \phi^* \Rightarrow \Delta \phi^* = -i\alpha \phi^*$

$$\alpha \Delta \phi^*$$

we note that: $\mathcal{L}' = |\partial_\mu (e^{i\alpha} \phi)|^2 - m^2 |e^{i\alpha} \phi|^2 = |\partial_\mu \phi|^2 - m^2 |\phi|^2 = \mathcal{L}$
 $[+\partial_\mu (J^\mu = 0)]$

Noether Theorem / current is

$$\begin{aligned}J^\mu &= \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) \Delta \phi + \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^*)} \right) \Delta \phi^* \\ &= (\partial_\mu \phi^*) i\alpha + (\partial_\mu \phi) (-i\alpha \phi^*) \\ &= i [(\partial_\mu \phi^*) \phi - (\partial_\mu \phi) \phi^*]\end{aligned}$$

J^μ is a conserved current

thus $\partial_\mu [(\partial_\mu \phi^*) \phi - (\partial_\mu \phi) \phi^*] = 0$

example 3: translations in space-time

Consider the translation/^{shift} of the space-time coordinates

$$x^\mu \rightarrow x^{\mu'} = x^\mu + a^\mu \quad a^\mu = \text{very small}$$

[note: change of origin is ~~$f(x) \rightarrow f(x-a)$~~] \sim infinitesimal

the transformation of the field is given by

$$\phi(x) \rightarrow \phi'(x) = \phi(x+a^\mu) = \phi(x) + \left(\frac{\partial \phi(x)}{\partial x^\mu} \right) a^\mu$$

$$= \phi + a^\mu \frac{\partial \phi}{\partial x^\mu}$$

\Rightarrow Infinitesimal translations are continuous transformations. small infinitesimal

We can apply the same approach to \mathcal{L} :

$$\mathcal{L} \rightarrow \mathcal{L}' = \mathcal{L} + \left(\frac{\partial \mathcal{L}}{\partial x^\mu} \right) a^\mu$$

$$= \mathcal{L} + a^\mu \frac{\partial \mathcal{L}}{\partial x^\mu}$$

$$= \mathcal{L} + a^\nu \underbrace{\frac{\partial \mathcal{L}}{\partial x^\nu} (\delta_\nu^\mu \mathcal{L})}_{\alpha \Delta \mathcal{L}' = \alpha \partial_\mu J^\mu}$$

$$\alpha \Delta \mathcal{L}' = \alpha \partial_\mu J^\mu$$

$$\text{so } J_\nu^\mu = \mathcal{L} S_\nu^\mu$$



$$S = \int d^4x \mathcal{L}, \text{ but } S' = \int d^4x' \mathcal{L}'$$

$$d^4x' = [\text{Jacobian}] d^4x$$

$$\text{for translations } d^4x' = d^4x$$

(but not for rotations or Lorentz boosts)

Noether current: $J_\nu^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} S_\nu^\mu - J_\nu^\mu$

$$J^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} (\partial_\nu \phi) - \mathcal{L} \delta^\mu_\nu$$

$$\equiv T^\mu_\nu = \text{stress-energy tensor}$$

$$\boxed{\partial_\mu T^\mu_\nu = 0}$$

continuity eq.
conservation law

or
energy-momentum tensor

What is the conserved "charge"?

→ not one "charge",
but 4 "charges".

$$Q_\nu = \int \int_\nu^0 d^3x = \text{invariant}$$

all space

Invariant for time translation:

$$Q_0 = \text{Hamiltonian} = \int \int_0^0 d^3x = \int \mathcal{E} d^3x \quad \left| \begin{array}{l} H = \nu = 0 \\ \frac{\partial \mathcal{L}}{\partial(\frac{\partial \phi}{\partial t})} - \mathcal{L} \\ \frac{\partial(\frac{\partial \phi}{\partial t})}{\partial t} = 0 \\ = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \mathcal{L} \\ = "p\dot{q} - \mathcal{L}" \end{array} \right.$$

Invariants for spatial translations:

$$Q_i = \text{momentum} = \int \int_{i=1,2,3}^0 d^3x$$

$$\text{physical momentum} = \text{conserved} = \int \frac{\partial \mathcal{L}}{\partial(\partial_i \phi)} d^3x \quad \left[\begin{array}{l} -\mathcal{L} \delta_{i=1,2,3}^0 \\ = 0 \end{array} \right]$$

canonical momentum density

back to E&M

Electromagnetic stress-Energy tensor $\phi \rightarrow A^\mu$

[Zangwill 24.4.3]

$$T^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\alpha)} \partial_\nu A^\alpha - \mathcal{L} \delta^\mu_\nu$$

and $\int T^\mu_\nu d^3x = \text{invariants}$ sum over field components

conservation law:
 $\partial_\mu T^\mu_\nu = 0$

stopped here

note: justification for sum over field components

$$\Delta \mathcal{L}' = \frac{\partial \mathcal{L}}{\partial \phi} \Delta \phi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta(\partial_\mu \phi) \underbrace{\left[+ \frac{\partial \mathcal{L}}{\partial \psi} \Delta \psi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi)} \delta(\partial_\mu \psi) \right]}_{\text{variation of 2nd field } \psi(x)}$$

Let's calculate T_{ν}^{μ} for $\mathcal{L}_{\text{EM}} = -\frac{1}{4\mu_0} F_{\alpha\beta} F^{\alpha\beta}$

$$\frac{\partial \mathcal{L}_{\text{EM}}}{\partial (\partial_\mu A^\alpha)} = -\frac{1}{4\mu_0} \frac{\partial}{\partial (\partial_\mu A^\alpha)} \left[[\partial_\alpha A_\beta - \partial_\beta A_\alpha] [\partial^\alpha A^\beta - \partial^\beta A^\alpha] \right]$$

$$= -\frac{1}{4\mu_0} \frac{\partial}{\partial (\partial_\mu A^\alpha)} \left\{ [g_{\alpha\beta} \partial_\alpha A^\beta - g_{\beta\alpha} \partial_\beta A^\alpha] \right. \\ \left. [g^{\lambda\sigma} \partial_\lambda A^\beta - g^{\beta\lambda} \partial_\beta A^\lambda] \right\}$$

$$= -\frac{1}{4\mu_0} \left\{ \underbrace{g_{12} \delta_0^\mu \delta_2^\lambda F^{\sigma\gamma}}_{F_\alpha^\mu} - \underbrace{g_{20} \delta_2^\mu \delta_2^\beta F^{\sigma\gamma}}_{F_\alpha^\mu} + \underbrace{F_{02}^\mu g^{\lambda\sigma} \delta_\lambda^\mu \delta_\lambda^\gamma}_{F_\alpha^\mu} \right. \\ \left. - F_{02}^\mu g^{\beta\gamma} \delta_\beta^\mu \delta_\beta^\sigma \right\}$$

$$= -\frac{1}{4\mu_0} \left\{ F_{12}^\mu + F_{20}^\mu + F_{02}^\mu + F_{02}^\mu \right\}$$

$$= -\frac{1}{\mu_0} F_{12}^\mu$$

$$F_{12}^\mu \\ = -F_{12}^\mu$$