

Tuesday, February 20, 2018

Stress-Energy tensor (continued)

Noether's theorem lead us to $\partial_\mu (\text{H})^{\mu\nu} = 0$ for no sources
 (charges & currents)
 with charges and currents, we get

$$\begin{aligned}\mu_0 \partial_\mu (\text{H})^{\mu\nu} &= \dots \text{ some algebra \& substitutions} \\ &\quad \text{source term} = \text{charges + currents} \\ &= -\mu_0 F^{\nu\alpha} J_\alpha + \underbrace{F_{\mu\lambda} (\partial^\mu F^{\nu\alpha})}_{\text{rename } \mu \rightarrow \sigma} - \frac{1}{2} F_{\alpha\gamma} (\partial^\nu F^{\alpha\gamma}) \\ &\quad \left. \begin{array}{l} \text{rename } \mu \rightarrow \sigma \\ \alpha \rightarrow \gamma \end{array} \right\} F_{\alpha\gamma} (\partial^\sigma F^{\nu\gamma}) \\ &= -\mu_0 F^{\nu\alpha} J_\alpha + \frac{1}{2} \left\{ F_{\alpha\gamma} (\partial^\sigma F^{\nu\gamma}) + \underbrace{F_{\alpha\gamma} (\partial^\nu F^{\sigma\gamma})}_{-F^{\sigma\nu}} - F_{\alpha\gamma} (\partial^\sigma F^{\sigma\gamma}) \right\} \\ &\quad \left. \begin{array}{l} \text{rename } \sigma \rightarrow \gamma \\ \gamma \rightarrow \sigma \end{array} \right\} F_{\gamma\sigma} (\partial^\sigma F^{\nu\sigma}) \\ &= -\mu_0 F^{\nu\alpha} J_\alpha - \frac{1}{2} F_{\alpha\gamma} \left\{ \underbrace{\partial^\sigma F^{\nu\gamma} + \partial^\nu F^{\sigma\gamma} + \partial^\sigma F^{\sigma\gamma}}_{=0 \text{ not obvious}} \right\}\end{aligned}$$

↳ see problem set #5
 problem N°1

↳ equivalent to $\partial_\mu G^{\mu\nu} = 0$
 i.e. Faraday's law & no magnetic monopoles law

$$\Rightarrow \boxed{\partial_\mu (\text{H})^{\mu\nu} = -F^{\nu\alpha} J_\alpha} \\ = J_\alpha F^{\alpha\nu}$$

Compare to
 Lorentz Force law
 in 4-vector notation

with sources, i.e. charges + currents

Conservation of Mechanical + EM
 Energy-momentum

Characteristics/Advantages of the Lagrangian Electrodynamics (i.e. field theory) approach:

- manifestly Lorentz invariant
- A^μ 4-potential obeys a least action principle
 - ↳ potentials are essential
- Fields are not so so different from particles.
- Space & time are on an equal footing
- More compact more unified description of EM
- physics is encoded in the formalism → easy to lose track of physics
- ↳ good for pushing theory further
- ↳ less useful for calculating actual physical systems.

the formalism → easy to lose track of physics

\vec{E} & \vec{B} are real, A^μ much less so

↳ just end up with lots of math/algebra

local theory: $\mathcal{L} \propto \partial_\nu A^\mu(x_0) \partial^\nu A^\mu(x_0)$
 not $\partial_\nu A^\mu(x_0) \partial^\nu A^\mu(x_1)$

↳ local conservation laws

- Continuous symmetries \leftrightarrow conserved charges & currents (Noether's theorem)
- ↳ gauge symmetry \leftrightarrow charge conservation
- Canonical momentum \rightarrow path/recipe for quantization

not \vec{E} & \vec{B} directly $\rightarrow A^\mu$ is quantized

$$[A^\mu(x^\nu), \frac{\partial \mathcal{L}(x^\nu)}{\partial (x^\nu)}] = i\hbar \delta(x^\nu - x^{\nu'})$$

long term objective: Equation of motion for spin in an EM field.
 (#3
 (travelling)

particle with an intrinsic magnetic moment

The Lorentz group

(Jackson chpt 11.7)
 (Goldstein 7.9)

Reminder: generators of transformations

Examples from QM:

generator of time evolution:
 (translation)
 ↳ Hamiltonian operator

$$U(t, t_0) |\psi\rangle = e^{-i \frac{H}{\hbar} (t - t_0)} |\psi(t)\rangle$$

generator of spatial translations
 ↳ momentum operator

$$|\psi(\vec{r} - \vec{r}_0)\rangle = e^{-i \frac{\vec{p} \cdot \vec{r}_0}{\hbar}} |\psi(\vec{r})\rangle$$

generator of rotations

$$|\psi(\vec{\phi})\rangle = e^{-i \frac{\vec{J} \cdot \vec{\Phi}}{\hbar}} |\psi\rangle$$

↳ angular momentum

$$\text{where } \vec{\Phi} = (\Phi_x, \Phi_y, \Phi_z)$$

Short term objective: get the generators of rotations for observables (not wavefunctions)

consider a rotation by $\phi = \phi_z$ around the z -axis for a spin- $\frac{1}{2}$ system

$$|\psi(\phi)\rangle = e^{-i \frac{S_z \phi}{\hbar}} |\psi\rangle$$

spinor = $\alpha | \uparrow \rangle + \beta | \downarrow \rangle$

the observable rotation is given by

$$\langle S_x \rangle_\phi = \langle \psi(\phi) | S_x | \psi(\phi) \rangle = \langle \psi | e^{i \frac{S_z \phi}{\hbar}} S_x e^{-i \frac{S_z \phi}{\hbar}} |\psi\rangle$$

$$= 1 + \left(\frac{-i\phi}{\hbar} \right) S_x + \frac{1}{2!} \left(\frac{-i\phi}{\hbar} \right)^2 S_x^2 + \frac{1}{3!} \left(\frac{-i\phi}{\hbar} \right)^3 S_x^3 + \dots$$

reminder: $S_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$= \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{+i\phi/2} \end{pmatrix}$$

$$\text{thus } \langle S_x \rangle_\phi = \langle 4 | \underbrace{\begin{pmatrix} e^{+it\sigma_z} & 0 \\ 0 & e^{-it\sigma_z} \end{pmatrix}}_{S_x} \frac{t}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \underbrace{\begin{pmatrix} e^{-it\sigma_z} & 0 \\ 0 & e^{+it\sigma_z} \end{pmatrix}}_{S_y} | 4 \rangle$$

\downarrow

$$\begin{pmatrix} 0 & e^{it\sigma_z} \\ e^{-it\sigma_z} & 0 \end{pmatrix}$$

$$\cos \phi \frac{t}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \sin(\phi) \frac{t}{2} \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} = \frac{t}{2} \begin{pmatrix} 0 & e^{it\sigma_z} \\ e^{-it\sigma_z} & 0 \end{pmatrix}$$

\leftarrow

$$\Rightarrow \langle S_x \rangle_\phi = \cos \phi \langle S_x \rangle + \sin \phi \langle S_y \rangle$$

(likewise, we can calculate $\langle S_y \rangle_\phi$ [alternate method] recall:

$$\langle S_y \rangle_\phi = \langle 4 | \underbrace{e^{\frac{i\phi}{\hbar} S_z}}_{S_x} e^{-\frac{i\phi}{\hbar} S_z} | 4 \rangle$$

$$S_y + \left(\frac{i\phi}{\hbar} \right) \underbrace{[S_z, S_y]}_{-i\hbar S_x} + \frac{1}{2!} \left(\frac{i\phi}{\hbar} \right)^2 \underbrace{[S_z, [S_z, S_y]]}_{-i\hbar S_x} + \dots$$

$$(i\hbar)(-i\hbar) S_y = \frac{i^2 \hbar^2}{2!} S_y$$

Baker-Hausdorff lemma

$$\begin{aligned} [S_x, S_y] &= i\hbar S_z \\ [S_z, S_x] &= i\hbar S_y \\ [S_y, S_z] &= i\hbar S_x \\ [S_i, S_j] &= i\hbar \epsilon_{ijk} S_k \quad \text{Levi-Civita symbol} \end{aligned}$$

$$+ \frac{1}{3!} \left(\frac{i\phi}{\hbar} \right)^3 \underbrace{[S_z, [S_z, [S_z, S_y]]]}_{\frac{i^3 \hbar^3}{3!} S_y} + \dots$$

$$(-i\hbar)(i\hbar)^2 S_y = -i\hbar^3 S_x$$

$$= \langle 4 | S_y \underbrace{\left\{ 1 - \frac{1}{2!} \phi^2 + \dots \right\}}_{\cos \phi} + S_x \underbrace{\left\{ \phi - \frac{1}{3!} \phi^3 + \dots \right\}}_{\sin \phi} | 4 \rangle$$

note: we only used the commutation relations
 \hookrightarrow generalizable to any angular momentum J i.e. spin - N

$$\Leftrightarrow \langle S_y \rangle_\phi = \cos \phi \langle S_y \rangle + \sin \phi \langle S_x \rangle$$

thus $\begin{pmatrix} \langle S_x \rangle_\phi \\ \langle S_y \rangle_\phi \end{pmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{pmatrix} \langle S_x \rangle \\ \langle S_y \rangle \end{pmatrix}$

standard rotation matrix/transformation

Generators of rotations for observables

consider the matrix $S_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$(S_3)^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$(S_3)^3 = S_3 (S_3)^2 = -S_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$(S_3)^4 = (S_3)^2 (S_3)^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow (S_3)^5 = S_3$$

ϕS_3

thus $e^{\phi S_3} = 1 + \phi S_3 + \frac{1}{2!} \phi^2 (S_3)^2 + \frac{1}{3!} \phi^3 (S_3)^3 + \frac{1}{4!} \phi^4 (S_3)^4 + \dots$

$$= \begin{bmatrix} 1 - \frac{1}{2!} \phi^2 + \frac{1}{4!} \phi^4 & -\phi + \frac{1}{3!} \phi^3 - \frac{1}{5!} \phi^5 + \dots \\ \phi - \frac{1}{3!} \phi^3 + \frac{1}{5!} \phi^5 + \dots & 1 - \frac{1}{2!} \phi^2 + \frac{1}{4!} \phi^4 - \dots \end{bmatrix}$$

$$= \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

In 3D, there are 3 generators of physical rotations [Jackson 11.7]

"rotate around x-axis": $S_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$, $S_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$

"rotate around z-axis": $S_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

vector of scalars $\vec{\phi} \cdot \vec{S}$ *vector of matrices*

$$R[\vec{\phi} = (\phi_x, \phi_y, \phi_z)] = e^{-\vec{\phi} \cdot \vec{S}} = \text{rotates a vector by angles } \vec{\phi}$$

$$R[\vec{\phi}] = e^{-\vec{\phi} \cdot \vec{S}} = \text{rotates a coordinate system by angles } \vec{\phi}.$$

In 4D Minkowski space, the 3 generators of physical rotations are

$$S_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & S_x & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; S_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & S_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$S_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\text{and } R[\vec{\phi}] = e^{-\vec{\phi} \cdot \vec{S}} = \text{rotates a reference frame by angles } \vec{\phi}.$$

note: $[S_i, S_j] = \epsilon_{ijk} S_k \Rightarrow \text{rotations in 3D form a group}$

↑ no "i"

SKIPPED
WEBSITE

Generators of Lorentz boosts for observables

Consider the matrix $k_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$(k_x)^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow (k_x)^3 = k_x$$

thus $e^{\gamma k_x} = 1 + \gamma k_x + \frac{1}{2!} \gamma^2 k_x^2 + \frac{1}{3!} \gamma^3 k_x^3 + \frac{1}{4!} \gamma^4 k_x^4 + \frac{1}{5!} \gamma^5 k_x^5 + \dots$

$$= \begin{bmatrix} 1 + \frac{1}{2!} \gamma^2 + \frac{1}{4!} \gamma^4 + \dots & \gamma + \frac{1}{3!} \gamma^3 + \frac{1}{5!} \gamma^5 + \dots \\ \cancel{\cosh \gamma} & \sinh \gamma \\ \gamma + \frac{1}{3!} \gamma^3 + \frac{1}{5!} \gamma^5 + \dots & 1 + \frac{1}{2!} \gamma^2 + \frac{1}{4!} \gamma^4 + \dots \end{bmatrix}$$

result: $\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta \\ \cosh \gamma & -\sinh \gamma \\ -\sinh \gamma & \cosh \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$ where $\tanh(\gamma) = \beta$

[January 23]
[Lecture]

$$e^{\gamma k_x} \quad (\gamma = \text{rapidity})$$

In 4D Minkowski space, the 3 generators of physical Lorentz boosts are

$$K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Thus $\Delta(\vec{\gamma} = (\gamma_1, \gamma_2, \gamma_3)) = e^{-\vec{\gamma} \cdot \vec{K}}$

line $\vec{\mu}$ \uparrow
column \uparrow
 $\vec{\gamma} \cdot \vec{K}$
vector of scalars
vector of matrices = Lorentz transformation with rapidities

note:

$$[K_i, K_j] = -\epsilon_{ijk} S_k$$

$$[S_i, K_j] = \epsilon_{ijk} K_k$$

! successive perpendicular Lorentz boosts give rise to a rotation of the inertial frame

Lorentz boosts and 3D rotations form a group.

More generally a rotation & Lorentz boost transformation is given by

$$A(\vec{\eta}, \vec{\phi}) = e^{-\vec{\phi} \cdot \vec{S} - \vec{\eta} \cdot \vec{K}}$$