

Generators of Lorentz boosts for observables [Jackson 11.7]

Consider the matrix $k_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$(k_x)^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow (k_x)^3 = k_x$$

thus $e^{\gamma k_x} = 1 + \gamma k_x + \frac{1}{2!} \gamma^2 k_x^2 + \frac{1}{3!} \gamma^3 k_x^3 + \frac{1}{4!} \gamma^4 k_x^4 + \frac{1}{5!} \gamma^5 k_x^5$

$$= \left[\underbrace{1 + \frac{1}{2!} \gamma^2 + \frac{1}{4!} \gamma^4 + \dots}_{\cosh \gamma} \quad \underbrace{\gamma + \frac{1}{3!} \gamma^3 + \frac{1}{5!} \gamma^5 + \dots}_{\sinh \gamma} \right]$$

$$\left[\underbrace{\gamma + \frac{1}{3!} \gamma^3 + \frac{1}{5!} \gamma^5 + \dots}_{\sinh \gamma} \quad \underbrace{1 + \frac{1}{2!} \gamma^2 + \frac{1}{4!} \gamma^4 + \dots}_{\cosh \gamma} \right]$$

recall: $\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma \beta \\ \cosh \gamma & -\sinh \gamma \\ -\sinh \gamma & \cosh \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$ where $\tanh(\eta) = \frac{v}{c}$

[Thursday, January 23] Lecture

($\eta = \text{rapidity or boost parameters}$)

In 4D Minkowski space, the 3 generators of physical Lorentz boosts are

$$K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Thus $\Delta(\vec{\eta} = (\eta_1, \eta_2, \eta_3))^\mu_\nu = e^{-\vec{\eta} \cdot \vec{K}}$

vector of scalar boost parameters

vector of matrices

= Lorentz transformation (rapidity format)

#2

rotation
generator
(not spin)



successive
perpendicular
Lorentz boosts
give rise to
a rotation
of the inertial
frame

note:

$$[K_i, K_j] = -\epsilon_{ijk} S_k \Rightarrow$$

$$[S_i, K_j] = \epsilon_{ijk} K_k$$

\Rightarrow Lorentz boosts and 3D rotations form a group.

More generally a rotation & Lorentz boost transformation is given by

$$A(\vec{\eta}, \vec{\phi}) = e^{-\vec{\eta} \cdot \vec{S} - \vec{\eta} \cdot \vec{K}}$$

Spin-orbit coupling in hydrogen

The B -field $\vec{B}_{e^-} = -\frac{1}{c^2} \vec{v} \times \gamma \vec{E}_{e^-}$ in the e^- 's instantaneous reference frame couples to e^- 's magnetic moment and causes the spin to precess: $\frac{d\vec{S}}{dt}|_{e^- \text{ frame}} = \vec{M}_e \times \vec{B}_{e^-}$ some work | in frame of e^-

in e^- 's frame So we expect $H_{\text{int}} = H^{\text{"spin-orbit"}} = -\vec{M}_e \cdot \vec{B}_{e^-}$

$$\begin{aligned}
 \text{note: } \vec{M}_e &= \vec{\mu} & g &= 2.002319304\dots \\
 &= \frac{g e \vec{S}}{2 m_e} & & \\
 &= -\frac{g e}{m_e} \vec{S} \cdot \left(-\frac{1}{c^2}\right) \vec{v} \times \gamma \vec{E}_L & & \\
 &= -\frac{(g e)^2}{4\pi\epsilon_0 m_e c^2} \frac{\gamma}{r^3} \vec{S} \cdot (\vec{v} \times \vec{r}) & & \frac{4\pi\epsilon_0}{F} \frac{r^2}{r^3} \\
 &= \frac{(g e)^2}{4\pi\epsilon_0 m_e^2 c^2} \frac{1}{r^3} \vec{l} \cdot \vec{S} & & \\
 \text{to lowest order in } \frac{v}{c}, r \rightarrow 1 & \approx \frac{(g e)^2}{4\pi\epsilon_0 m_e^2 c^2} \frac{1}{r^3} \vec{l} \cdot \vec{S} & & = H_{\text{Beff}}
 \end{aligned}$$

However, experiment shows that

$$H_{\text{spin-orbit}} = \frac{1}{2} \frac{(g e)^2}{4\pi\epsilon_0 m_e^2 c^2} \frac{1}{r^3} \vec{l} \cdot \vec{S}$$

accurate to 1st order in $\frac{v}{c}$

due to Thomas precession

↳ Dirac equation also reproduces this result (automatically)

Thomas Precession

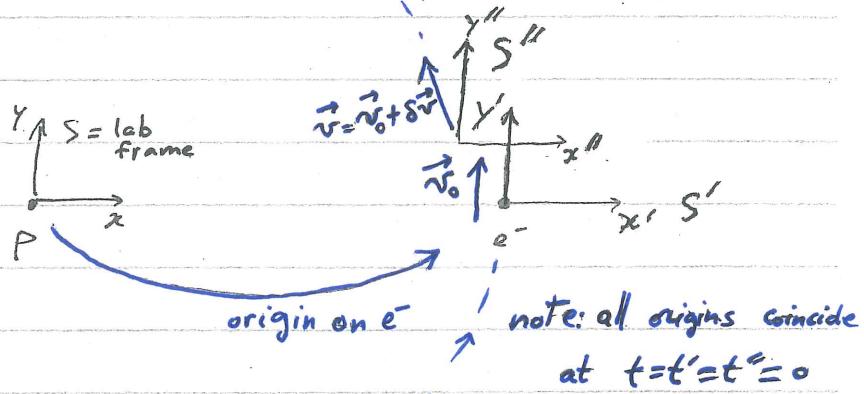
This is a purely, relativistic kinematic effect due to the rotating nature of the e^- 's instantaneous reference frame.

It originates from the fact that the result of 2 Lorentz boosts in different directions can only be written as a single Lorentz boost combined with a rotation.

Strategy: derive precession \rightarrow modify spin-orbit coupling Hamiltonian #4

Mathematical Derivation

[Jackson 11.8]



$$\tanh(\gamma_0) = \beta_0 = \frac{v_0}{c}$$

$$\tanh(\gamma) = \beta = \frac{v}{c} = \sqrt{\left(v_0 + \delta v_x\right)^2 + \delta v_z^2} \Rightarrow \gamma = \tanh^{-1}(\beta)$$

Coordinate Transformations:

$$r^\nu = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

$$r^{\mu'} = \Delta(\vec{\beta}_0)^\mu{}_\nu r^\nu \Leftrightarrow r^\nu = \Delta(-\vec{\beta}_0)^\nu{}_\mu r^{\mu'}$$

$$\text{also } r^{\mu''} = \Delta(\vec{\beta})^\mu{}_\nu r^\nu \Leftrightarrow$$

$$\Rightarrow r^{\mu''} = \Delta(\vec{\beta} = \vec{\beta}_0 + \delta\vec{\beta})^\mu{}_\nu \underbrace{\Delta(-\vec{\beta}_0)_\nu{}^\nu}_{\Delta(\vec{\beta}_0)^{-1}} r^{\nu'} =$$

transformation from e^- 's

$$\xrightarrow{\quad} A(\Delta\vec{\Phi}, \Delta\vec{\eta}) ?$$

rest frame to the next at time $t + \delta t$

$$= e^{-\Delta\vec{\Phi} \cdot \vec{S} - \Delta\vec{\eta} \cdot \vec{K}}$$

boost generator
(not spin)
Rotation generator

note: $\vec{\gamma} = \left[\frac{(\beta_0 + \delta\beta_{11})}{\beta} \hat{y} + \frac{\delta\beta_{12}\hat{x}}{\beta} \right] \tanh^{-1}(\beta)$ since $\vec{v}_0 = v_0 \hat{y}$
and $\vec{\gamma} \parallel \vec{v}$

$$\hat{v} = \hat{\beta}$$

$$\Delta(\vec{\beta}) \Delta(-\vec{\beta}_0) = \exp \left\{ \left[\frac{(\beta_0 + \delta\beta_{11})}{\beta} K_y + \frac{\delta\beta_{12} K_x}{\beta} \right] \tanh^{-1}(\beta) \right\} \times$$

$$\exp \left\{ f \left(+ \tanh^{-1}(\beta_0) K_y \right) \right\}$$

Recall Baker-Campbell-Hausdorff formulae:

$A \& B$ are matrices

$$A = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ C & D \end{pmatrix}$$

$$A + B = \frac{1}{2} [A, B] + \frac{1}{12} ([A, [A, B]] + [B, [B, A]]) + \dots$$

thus $\Delta(\vec{\beta} = \vec{\beta}_0 + \delta\vec{\beta}) \Delta(-\vec{\beta}_0)$

$$= \exp \left\{ - \left[\frac{(\beta_0 + \delta\beta_{||}) \tanh^{-1}(\beta) - \tanh^{-1}(\beta_0)}{\beta} \right] K_y - \frac{\delta\beta_{\perp}}{\beta} \tanh^{-1}(\beta) K_x \right\}$$

non-relativistic limit: $\beta_0, \beta \ll 1 \Rightarrow \tanh^{-1}(\beta) = \beta$

↳ first order in $\delta\beta$

$$+ \frac{1}{2} \left[- \left(\frac{\beta_0 + \delta\beta_{||}}{\beta} \right) \tanh^{-1}(\beta) \tanh^{-1}(\beta_0) [K_y, K_y] \right] \underset{=0}{\cancel{=}}$$

recall: $[K_i, K_j] = -\epsilon_{ijk} S_k$

$$- \frac{\delta\beta_{\perp}}{\beta} \tanh^{-1}(\beta) \tanh^{-1}(\beta_0) [K_x, K_y] + \dots$$

$$= \exp \left\{ - \delta\beta_{||} K_y - \delta\beta_{\perp} K_x + \frac{1}{2} \delta\beta_{\perp} \beta_0 S_3 + \dots \right\}$$

1st order in $\delta\beta$ ↑ rotation generator

$$= \exp \left(- \delta\vec{\beta} \cdot \vec{K} - \Delta\phi S_3 \right) \approx \exp(-\delta\vec{\beta} \cdot \vec{K}) \exp(-\Delta\phi S_3)$$

$$\text{with } \Delta\phi = -\frac{1}{2} \delta\beta_{\perp} \beta_0$$

↳ since $\delta\beta_{\perp} < 0 \Rightarrow \Delta\phi > 0$

pure rotation by angle $\Delta\phi$ around z -axis

rotation rate of e^- 's rest frames' with respect to itself is

thus

$$\omega''_{\text{Thomas}} = \omega'_{\text{Thomas}} = \frac{\Delta\phi}{\Delta t} = \lim_{\Delta t \rightarrow 0} \left(\frac{1}{2} \frac{\delta\beta_{\perp}}{\delta t} \beta_0 \right) = -\frac{1}{2} \frac{1}{c^2} \frac{dv_1}{dt} v_0$$

$$\frac{\Delta\phi}{\Delta t} = \frac{\delta\beta_{\perp}}{\delta t}$$

$\gamma \rightarrow 1$
in non-relativistic
limit

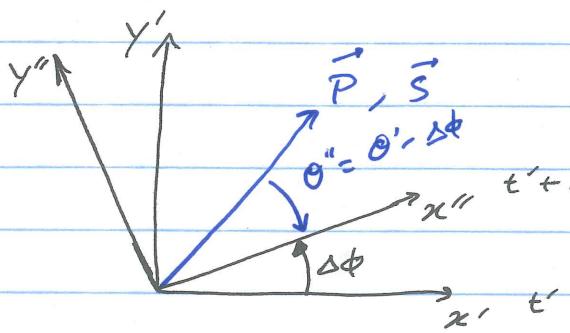
$$\Rightarrow \omega'_{\text{Thomas}} = -\frac{1}{2} \frac{av_0}{c^2}$$

$$\vec{\omega}'_{\text{Thomas}} = +\frac{1}{2} \frac{\vec{v} \times \vec{a}}{c^2}$$

note: $\frac{dv_1}{dt} = a < 0$

$\Rightarrow \omega'_{\text{Thomas}} > 0$
↳ points into \vec{z} direction

Consider a "pointer" vector \vec{P} (or spin \vec{S}) on the e^-



$$\begin{aligned} \vec{r}_p &= \exp[-\delta\phi S_y] \vec{r}'_p \\ &= \begin{bmatrix} \cos\delta\phi & \sin\delta\phi & 0 \\ -\sin\delta\phi & \cos\delta\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x'_p \\ y'_p \\ z'_p \end{pmatrix} \end{aligned}$$

In the frame of the e^- , it looks as though \vec{P} is precessing clockwise.

Stopped here

Rotational motion reminder

A vector \vec{P} in counterclockwise circular motion has the equation of motion with rotation speed/frequency vector $\vec{\omega} = \nu \hat{z}$

$$\frac{d\vec{P}}{dt} = \vec{\omega} \times \vec{P}$$

In the frame of the e^- , \vec{S} is precessing clockwise, so

$$\left. \frac{d\vec{S}}{dt} \right|_{e^- \text{ frame}} = -\vec{\omega}'_{\text{Thomas}} \times \vec{S} = \vec{S} \times \vec{\omega}'_{\text{Thomas}}$$

If we include spin precession due to \vec{B}_{e^-} in the e^- 's frame, then

$$\left. \frac{d\vec{S}}{dt} \right|_{e^- \text{ frame}} = \vec{S} \times \vec{\omega}'_{\text{Thomas}} + \vec{S} \times \frac{q_e}{m_e} \vec{B}_{e^-}$$

$$\vec{M}_e = \frac{q_e}{m_e} \vec{S}$$

⇒ The Thomas precession just looks like it is due to some additional fictitious magnetic field (in e^- frame).

→ $H_{\text{Thomas}} = -\vec{S} \cdot \vec{\omega}'_{\text{Thomas}}$

$$\tilde{\omega}'_{\text{Thomas}} = \frac{1}{2} \frac{\vec{v} \times \vec{a}}{c^2} \vec{F}/m_e$$

$$= \frac{1}{2} \frac{1}{c^2} \frac{m_e \vec{v}}{m_e} \times \frac{q_e E}{m_e}$$

$$= \frac{1}{2} \frac{1}{c^2} \cancel{\frac{m_e \vec{v}}{m_e}} \times \frac{1}{m_e} \frac{q_e |q_e|}{4\pi\epsilon_0} \left(\frac{\vec{r}}{r^3} \right) \vec{F}$$

$$= -\frac{1}{2} \frac{|q_e l|^2}{4\pi\epsilon_0} \frac{1}{m_e^2 c^2} \frac{1}{r^3} \cancel{\frac{\vec{l} \times \vec{r}}{-\vec{l}}}$$

$$= -\frac{1}{2} \frac{|q_e l|^2}{4\pi\epsilon_0} \frac{1}{m_e^2 c^2} \frac{1}{r^3} \vec{l}$$

$$\Rightarrow H_{\text{Thomas}} = -\frac{1}{2} \frac{|q_e l|^2}{4\pi\epsilon_0} \frac{1}{m_e^2 c^2} \frac{1}{r^3} \vec{l} \cdot \vec{s}$$

$$\text{thus } H_{\text{spin-orbit}} = H_{\text{Beff}} + H_{\text{Thomas}}$$

$$= \frac{|q_e l|^2}{4\pi\epsilon_0} \frac{1}{m_e^2 c^2} \frac{1}{r^3} \vec{l} \cdot \vec{s} - \frac{1}{2} \frac{|q_e l|^2}{4\pi\epsilon_0} \frac{1}{m_e^2 c^2} \frac{1}{r^3} \vec{l} \cdot \vec{s}$$

$$\Rightarrow \boxed{H_{\text{spin-orbit}} = \frac{1}{2} \frac{|q_e l|^2}{4\pi\epsilon_0} \frac{1}{m_e^2 c^2} \frac{1}{r^3} \vec{l} \cdot \vec{s}}$$

to lowest
order in $\frac{v}{c}$
(1st order)

⚠ Thomas precession also applies to JLab polarized e^- beam.