

Tuesday, March 27, 2018

#1

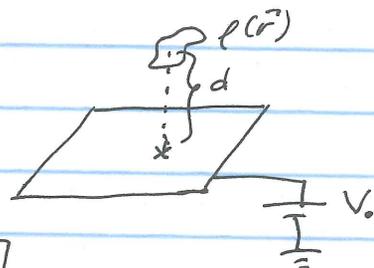
Green's functions (continued)

Last time, we saw that

$$V(\vec{r}) = - \int_V \underbrace{G_0(\vec{r}, \vec{r}')}_{\text{Dirichlet b.c.: } G_0=0 \text{ on boundary}} \frac{\rho(\vec{r}')}{\epsilon_0} d^3r' + \int_S \underbrace{V(\vec{r}')}_{\text{specifies b.c. on } S} \frac{\partial G_0(\vec{r}, \vec{r}')}{\partial n'}$$

Example:

Compute $V(\vec{r})$ and $G_0(\vec{r}, \vec{r}')$ for an infinite conducting plane at $z=0$ held at potential V_0 .



Last time, we showed that for $z > 0$:
(from method of images)

$$G_0(\vec{r}, \vec{r}') = -\frac{1}{4\pi} \left[\frac{1}{|\vec{r} - \vec{r}'|} - \frac{1}{|\vec{r} - \vec{r}'_{\text{image}}|} \right] \text{ with } \vec{r}'_{\text{image}} = (x', y', -z')$$

$$= -\frac{1}{4\pi} \left[\frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} - \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z+z')^2}} \right]$$

for the surface integral, we need to calculate

$$\frac{\partial G_0(\vec{r}, \vec{r}')}{\partial n'} = - \frac{\partial G_0}{\partial z'} \Big|_{z'=0}$$

since \hat{n}' points out of $V(z > 0)$, so $\hat{n}' = -\hat{z}'$ (i.e. points towards negative z')

$$= +\frac{1}{4\pi} \left\{ \left(\frac{1}{2} \right) \frac{z(z-z')(-1)}{[(x-x')^2 + (y-y')^2 + (z-z')^2]^{3/2}} + \left(\frac{1}{2} \right) \frac{z(z+z')(+1)}{[(x-x')^2 + (y-y')^2 + (z+z')^2]^{3/2}} \right\}_{z'=0}$$

$$= +\frac{1}{4\pi} \frac{2z}{[(x-x')^2 + (y-y')^2 + z^2]^{3/2}}$$

thus for $\rho(\vec{r}') = 0$, the Dirichlet Green's function formula gives

$$V(\vec{r}) = + \frac{1}{4\pi} \int V(\vec{r}') \frac{2z}{[(x-x')^2 + (y-y')^2 + z^2]^{3/2}} dx'dy'$$

$V(\vec{r}')$ on surface S
determines $V(\vec{r})$ in V

self-consistency check: does formula give $V(\vec{r})|_{\vec{r} \in S}$?

$$V(\vec{r} \in S) \stackrel{\text{i.e. } z \rightarrow 0^+}{=} \int V(\vec{r}') \lim_{z \rightarrow 0^+} \frac{1}{2\pi} \frac{z}{[(x-x')^2 + (y-y')^2 + z^2]^{3/2}} dx'dy'$$

$\delta(x-x') \delta(y-y')$

$$\Rightarrow V(\vec{r})|_{\vec{r} \in S} = V(\vec{r})|_{\vec{r} \in S} \rightarrow \text{internally consistent}$$

The general solution is thus

$$V(\vec{r} \in V) \Big|_{z > 0} = + \frac{1}{4\pi\epsilon_0} \int \rho(\vec{r}') \left[\frac{1}{|\vec{r} - \vec{r}'|} - \frac{1}{|\vec{r} - \vec{r}'_{\text{image}}|} \right] dx'dy'dz'$$

$(x', y', -z')$

Contribution from $\rho(\vec{r}')$

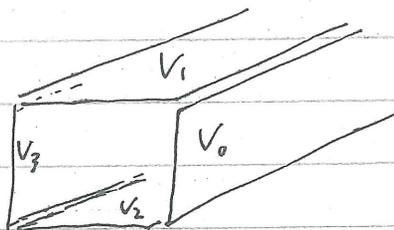
Contribution from surface

$$+ \int_S V_0 \frac{1}{2\pi} \frac{z}{[(x-x')^2 + (y-y')^2 + z^2]^{3/2}} dx'dy'$$

in principle, we could make this $V_0(x', y')$

Separation of variables

Consider a system with cartesian symmetry



rectangular duct
with conductor walls

Q: what is the potential inside the duct?

no charges (except on boundaries)

↳ Laplace's equation: $\nabla^2 V = 0$

$$\Leftrightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) V(x, y, z) = 0$$

only possible because of
symmetry

We will consider separable solutions of the form

$$V(x, y, z) = X(x) Y(y) Z(z)$$

divide by $X(x)Y(y)Z(z)$ ↓

$$\Rightarrow Y(y)Z(z) \frac{\partial^2 X}{\partial x^2} + X(x)Z(z) \frac{\partial^2 Y}{\partial y^2} + X(x)Y(y) \frac{\partial^2 Z}{\partial z^2} = 0$$

$$\Rightarrow \frac{1}{X(x)} \frac{\partial^2 X}{\partial x^2} + \frac{1}{Y(y)} \frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z(z)} \frac{\partial^2 Z}{\partial z^2} = 0$$

Since the solution must work for all (x, y, z) in the solution volume, then each term is equal to a constant such that

$$\frac{1}{X} \frac{\partial^2 X}{\partial x^2} = C_x \Leftrightarrow \frac{\partial^2 X}{\partial x^2} = C_x X = \alpha^2 X$$

$$\frac{1}{Y} \frac{\partial^2 Y}{\partial y^2} = C_y \Leftrightarrow \frac{\partial^2 Y}{\partial y^2} = C_y Y = \beta^2 Y$$

$$\frac{1}{Z} \frac{\partial^2 Z}{\partial z^2} = C_z \Leftrightarrow \frac{\partial^2 Z}{\partial z^2} = C_z Z = \gamma^2 Z$$

with

$$C_x + C_y + C_z = 0$$

$$\alpha^2 + \beta^2 + \gamma^2 = 0$$

$$\alpha, \beta, \gamma \in \mathbb{C}$$

Solutions:

$$X_\alpha = \begin{cases} A_0 + B_0 x & \alpha = 0 \leftarrow \text{homogeneous solution} \\ A_\alpha e^{\alpha x} + B_\alpha e^{-\alpha x} & \alpha \neq 0 \leftarrow \text{inhomogeneous solution} \end{cases}$$

$$Y_\beta = \begin{cases} C_0 + D_0 y & \beta = 0 \\ C_\beta e^{\beta y} + D_\beta e^{-\beta y} & \beta \neq 0 \end{cases}$$

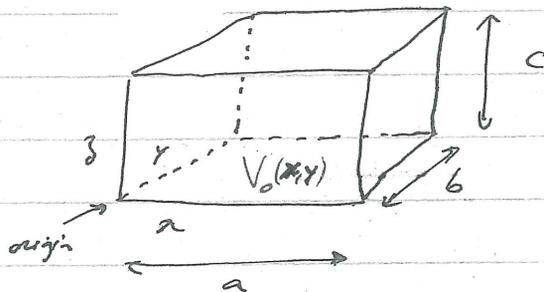
$$Z_\gamma = \begin{cases} E_0 + F_0 z & \gamma = 0 \\ E_\gamma e^{\gamma z} + F_\gamma e^{-\gamma z} & \gamma \neq 0 \end{cases}$$

general solution: we consider any linear superposition of these solutions.

#5

$$V(x, y, z) = \sum_{\alpha, \beta, \gamma} X_{\alpha}(x) Y_{\beta}(y) Z_{\gamma}(z) \delta(\underbrace{\alpha^2 + \beta^2 + \gamma^2}_{\text{enforces } \alpha^2 + \beta^2 + \gamma^2 = 0})$$

Example: Consider a rectangular conducting box with 5 sides at potential $V=0$ and one side at potential $V_0(x, y)$



boundary conditions

$$\begin{cases} X_{\alpha}(x=0) = 0 \\ X_{\alpha}(x=a) = 0 \end{cases} \quad \begin{cases} Y_{\beta}(y=0) = 0 \\ Y_{\beta}(y=b) = 0 \end{cases}$$

$$\begin{cases} Z_{\gamma}(z=0) = V_0(x, y) \\ Z_{\gamma}(z=c) = 0 \end{cases}$$

for X_{α} :

$$X_{\alpha}(x=0) = 0 \Rightarrow A_{\alpha} + B_{\alpha} = 0 \Leftrightarrow A_{\alpha} = -B_{\alpha}$$

$$X_{\alpha}(x=a) = 0 \Rightarrow A_{\alpha} e^{\alpha a} - A_{\alpha} e^{-\alpha a} = 0$$

$$\Rightarrow e^{\alpha a} = e^{-\alpha a}$$

$$\Leftrightarrow e^{2\alpha a} = 1 \Rightarrow \begin{cases} 2\alpha a = 0 \Rightarrow \alpha = 0 \\ \text{or} \\ 2\alpha a = n \cdot i \cdot 2\pi \end{cases}$$

$$\Rightarrow \boxed{\alpha = i \frac{n\pi}{a}}$$

$n = 0, \pm 1, \pm 2, \dots$

$$\Rightarrow X_{\alpha}(x) = 2i A_{\alpha} \left[\frac{e^{i(\frac{n\pi}{a})x} - e^{-i(\frac{n\pi}{a})x}}{2i} \right]$$

$$\Rightarrow \boxed{X_n(x) \equiv A_n \sin\left(\frac{n\pi x}{a}\right)}$$

for Y_{β} : similarly,

$$\boxed{Y_m(y) \equiv C_m \sin\left(\frac{m\pi y}{b}\right)}$$

and $\boxed{\beta = i \frac{m\pi}{b}}$

for Z_γ : $Z_\gamma(z=0) = V_0 \Rightarrow E_\gamma + F_\gamma = V_0(x,y) \rightarrow$ hard, deal with later

$$Z_\gamma(z=c) = 0 \Rightarrow E_\gamma e^{\gamma c} + F_\gamma e^{-\gamma c} = 0$$

$$\Rightarrow \begin{cases} E_\gamma = e^{-\gamma c} G_\gamma / 2 \\ F_\gamma = -e^{+\gamma c} G_\gamma / 2 \end{cases} \quad \left| \begin{array}{l} \text{without loss} \\ \text{of generality} \end{array} \right.$$

$$\Rightarrow Z_\gamma = G_\gamma \left[\frac{e^{-\gamma z} e^{\gamma c} - e^{+\gamma z} e^{-\gamma c}}{2} \right]$$

$$= G_\gamma \left[\frac{e^{\gamma(z-c)} - e^{-\gamma(z-c)}}{2} \right]$$

$$= G_\gamma \sinh[\gamma(z-c)]$$

$$\Rightarrow Z_\gamma(z) = G_\gamma \sinh[\gamma(z-c)] \quad \text{with } \gamma^2 = -(\alpha^2 + \beta^2)$$

$$= \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2$$

$$\Rightarrow \boxed{Z_{mn}(z) = G_{mn} \sinh[\gamma_{mn}(z-c)]}$$

$$\Rightarrow \boxed{\gamma_{mn} = \sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}}$$

Thus $V(x,y,z) = \sum_{m,n=1}^{\infty} \underbrace{G_{mn} A_n C_m}_{\text{rename } V_{mn}} \sin\left(\frac{n\pi}{a}x\right) \sin\left(\frac{m\pi}{b}y\right) \sinh[\gamma_{mn}(z-c)]$

final boundary condition: $V(x,y,z=0) = V_0(x,y)$

$$\Rightarrow V_0(x,y) = \sum_{m,n=1}^{\infty} V_{mn} \sin\left(\frac{n\pi}{a}x\right) \sin\left(\frac{m\pi}{b}y\right) \sinh[\gamma_{mn}(\overset{0}{z-c})]$$

$$= \sum_{m,n=1}^{\infty} V_{mn} \sinh(\gamma_{mn}c) \sin\left(\frac{n\pi}{a}x\right) \sin\left(\frac{m\pi}{b}y\right)$$

\Rightarrow this is a Fourier representation of $V_0(x,y)$
(series)