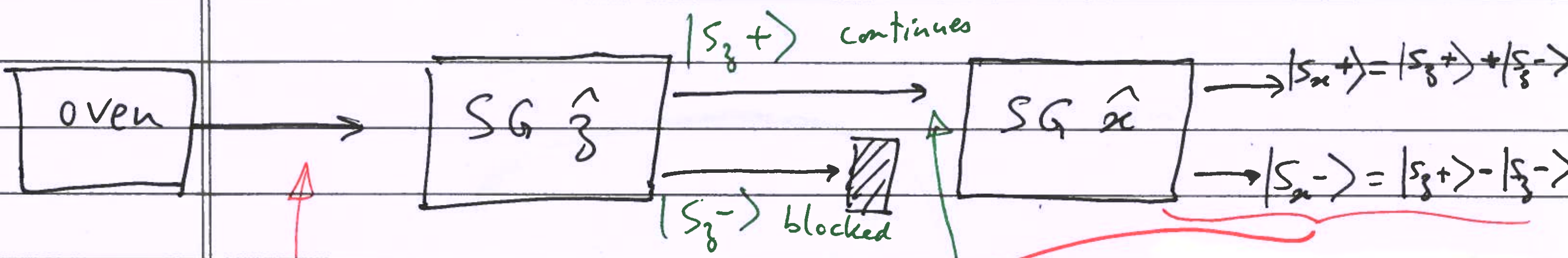
Tuesday, September 1, 2026

Sequential Stern-Gerlach selectors



statistical mixture of spin states (probably some residual quantum superposition)

" $|S_z+\rangle = |S_x+\rangle + |S_x-\rangle$ "

$\underbrace{|S_x+\rangle + |S_x-\rangle}_{\text{quantum superposition of spin states}}$

$|S_z+\rangle = \underbrace{|S_x+\rangle|\text{top}\rangle + |S_x-\rangle|\text{bottom}\rangle}$

quantum superposition of spin & position states

[Spin & position are entangled]

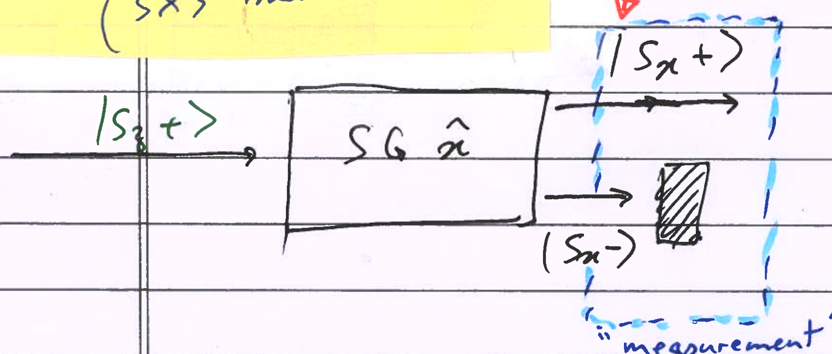
Quiz on Thursday, Sept. 3

covering

Eigenvalues & Eigenvectors

for 2×2 matrices

3×3 matrices



$|S_x+\rangle|\text{top}\rangle$

"measurement"

↳ superposition destroyed → collapse of superposition

Vector Spaces

A vector space $\mathbb{V}$ is a collection of elements/objects called vectors such that for $\underbrace{|v\rangle, |w\rangle, |z\rangle}$ $\in \mathbb{V}$ we have:

we use Dirac "ket" notation: $\vec{v} \equiv |v\rangle$

1) Commutative addition: $|v\rangle + |w\rangle = |w\rangle + |v\rangle \in \mathbb{V}$

2) Associative addition: $|v\rangle + (|w\rangle + |z\rangle) = (|v\rangle + |w\rangle) + |z\rangle \in \mathbb{V}$

3) There exists a unique "null vector" or "zero vector" $|0\rangle \in \mathbb{V}$ such that $|v\rangle + |0\rangle = |0\rangle + |v\rangle = |v\rangle$

note: "$|0\rangle$" in QM does not always denote the zero vector.

4) There exists a unique inverse under addition $|-v\rangle$ for every vector $|v\rangle$ such that $|v\rangle + |-v\rangle = |0\rangle$

5) Consider the scalar numbers $\alpha, \beta \in \mathbb{C}$, then $\alpha|v\rangle \in \mathbb{V}$ (for any $|v\rangle$) [also $\beta|v\rangle \in \mathbb{V}$]

6) Distributive law: $(\alpha + \beta)|v\rangle = \alpha|v\rangle + \beta|v\rangle$

7) $\alpha(|v\rangle + |w\rangle) = \alpha|v\rangle + \alpha|w\rangle$

8) $(\alpha\beta)|v\rangle = \alpha(\beta|v\rangle)$

9) $1 \cdot |v\rangle = |v\rangle$ (1 = identity scalar)

Linear dependence

Vectors $|v_1\rangle, |v_2\rangle, \ldots, |v_n\rangle$ are linearly dependent if there is a linear combination of them that gives $|0\rangle$:

There exist $\alpha_i \in \mathbb{C}$ such that $\sum_{i=1}^{n} \alpha_i |v_i\rangle = |0\rangle$
$(\alpha_i \neq 0)$

Note: A set of vectors is linearly independent if no such $\{\alpha_i\}$ exist.

Dimension of Vector Space

A vector space $\mathbb{V}$ is n-dimensional if there is a set of n linearly independent vectors $\{|v_1\rangle, |v_2\rangle, \ldots, |v_n\rangle\}$, but all sets of n+1 vectors are linearly dependent.

Basis: The set $\{|v_1\rangle, \ldots, |v_n\rangle\}$ is a basis for the vector space (not unique)

In this case any $|v\rangle \in \mathbb{V}$ can be written as

$$|v\rangle = \sum_{i=1}^{n} \alpha_i |v_i\rangle$$

[the set $\{\alpha_i\}$ is unique for a given $|v\rangle$]

Note: We will frequently work with infinite dimensional vector spaces called Hilbert spaces.

Subspace

$\mathbb{V}'$ is a subspace of $\mathbb{V}$ ($\mathbb{V}' \subset \mathbb{V}$) if for all $|v\rangle, |w\rangle \in \mathbb{V}'$ (and all $\alpha, \beta \in \mathbb{C}$), then $\alpha|v\rangle \in \mathbb{V}'$, $\beta|w\rangle$

and $\alpha|v\rangle + \beta|w\rangle \in \mathbb{V}'$ [$\dim(\mathbb{V}') \leq \dim(\mathbb{V})$]

Inner Product (aka "scalar product", "dot product")

Consider two vectors $|v\rangle, |w\rangle$, then their inner product $\langle w, v\rangle \in \mathbb{C}$ has the properties:

1) $\langle w|v\rangle = \langle v|w\rangle^*$ $\quad$ $[\,|v\rangle, |w\rangle, |u\rangle \in \mathbb{V}\,]$

2) $\langle w|\big(\alpha|v\rangle + \beta|u\rangle\big) = \alpha\langle w|v\rangle + \beta\langle w|u\rangle$

3) $\langle v|v\rangle \geqslant 0$, and $\langle v|v\rangle = 0 \iff |v\rangle = |0\rangle$

Clarification: If we write $\alpha|v\rangle = |\alpha v\rangle$, then
$\langle w|\alpha v\rangle = \langle w|(\alpha|v\rangle) = \alpha\langle w|v\rangle$

According to (1), we can write $\langle w|\underbrace{\alpha v}_{\text{"ket"}}\rangle = \langle\underbrace{\alpha v}_{\text{"bra"}}|w\rangle^*$

Q: Is $\langle \alpha v|w\rangle \stackrel{?}{=} \alpha\langle v|w\rangle$? NO

Answer: $\langle\alpha v|w\rangle = \langle w|\alpha v\rangle^* = \big(\langle w|\alpha v\rangle\big)^*$
$= \big(\alpha\langle w|v\rangle\big)^* = \alpha^*\langle w|v\rangle^*$
$= \alpha^*\langle v|w\rangle$

$\Rightarrow$ $\boxed{\langle\alpha v|w\rangle = \alpha^*\langle v|w\rangle}$

$$\boxed{\begin{array}{lrcl} \text{Thus} & |\alpha v\rangle &=& \alpha|v\rangle \\ \text{but} & \langle\alpha v| &=& \alpha^*\langle v| \end{array}}$$

$\longrightarrow$ bra behaves differently from ket

Orthogonality

Two vectors $|v\rangle, |w\rangle$ are said to be orthogonal iff

$$\langle v|w\rangle = 0$$
$$\Leftrightarrow \langle w|v\rangle = 0$$

note: $\langle v|0\rangle = \langle v|w-w\rangle$
$= \langle v|w\rangle - \langle v|w\rangle$
$= 0$

Norm: the norm of a vector (i.e. it's "length") is defined as

$$\boxed{\|v\| = \sqrt{\langle v|v\rangle}} \quad \Longleftrightarrow \quad \boxed{\|v\|^2 = \langle v|v\rangle}$$

"magnitude"

(Cauchy-) Schwarz Inequality: $\boxed{|\langle v|w\rangle|^2 \leq \|v\|^2 \|w\|^2}$

Orthonormality: 2 vectors $|v\rangle, |w\rangle$ are orthonormal if $\langle v|w\rangle = 0$ and $\|v\| = \|w\| = 1$.

Inner product with an orthonormal bases

Given a basis of orthonormal vectors $\{|i\rangle\}$, then any vectors $|w\rangle, |v\rangle$ can be written uniquely as

$$|v\rangle = \sum_i v_i |i\rangle \quad \text{and} \quad |w\rangle = \sum_j w_j |j\rangle \quad \text{with } v_i, w_j \in \mathbb{C}$$

Then
$$\langle v|w\rangle = \left(\sum_i v_i^* \langle i|\right)\left(\sum_j w_j |j\rangle\right)$$

$$= \sum_{i,j} v_i^* w_j \underbrace{\langle i|j\rangle}_{= \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}}$$

Kronecker Delta

$$\Rightarrow \langle v|w\rangle = \sum_i v_i^* w_i$$

Also $\|v\|^2 = \langle v|v\rangle = \sum_i |v_i|^2$

USEFUL TRICK

If $|j\rangle$ is one of the orthonormal basis vectors, then $\langle j|v\rangle = \sum_i v_i \langle j|i\rangle = \sum_i v_i \delta_{ij}$

$$= v_j$$

$$\Rightarrow \boxed{\langle j|v\rangle = v_j} \rightarrow \langle i|v\rangle$$

Importantly, we can write $|v\rangle = \sum_i v_i |i\rangle$

$$= \sum_i \langle i|v\rangle\, |i\rangle$$

$$\Rightarrow \quad |v\rangle = \underbrace{\sum_i |i\rangle\langle i|}_{\text{identity operator} = I = \mathbb{1}}\, |v\rangle$$

$$\Rightarrow \boxed{\text{Identity operator} = \mathbb{1} = \sum_i |i\rangle\langle i|}$$