Read Sakurai & Napolitano Sections 1.3, 1.5 and 1.4

Thursday, September 3, 2026

Example vector space: functions $f(x)$ on $x \in [0, 2\pi]$ or $x \in [-\pi, \pi]$

basis (i.e. Fourier basis) $= \left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos(nx), \frac{1}{\sqrt{\pi}} \sin(nx) \right\} = \{|i\rangle\}$
with $n = 1, 2, 3, \ldots$

Inner product: $\langle f(x), g(x) \rangle = \int_0^{2\pi} f^*(x)\, g(x)\, dx$ ($\int_{-\pi}^{+\pi}$), note: $\langle i|j\rangle = \delta_{ij}$ $\Rightarrow$ orthonormal basis

USEFUL TRICK

If $|j\rangle$ is one of the orthonormal basis vectors, then $\langle j|v\rangle = \sum_i v_i \langle j|i\rangle = \sum_i v_i \delta_{ij}$

$|v\rangle = \sum_i v_i |i\rangle$

$= v_j$

$$\Rightarrow \boxed{\langle j|v\rangle = v_j}$$

$\langle i|v\rangle$

Importantly, we can write $|v\rangle = \sum_i v_i |i\rangle$

$= \sum_i \langle i|v\rangle\, |i\rangle$

$$\Rightarrow |v\rangle = \underbrace{\sum_i |i\rangle\langle i|}_{\text{identity operator} = I = \mathbb{1}}\, v\rangle$$

$$\Rightarrow \boxed{\text{Identity operator} = \mathbb{1} = \sum_i |i\rangle\langle i|}$$

<u>Projection operator</u>: Pick out the coefficient (component) of a particular basis vector.

$$P_i = |i\rangle\langle i|$$

Thus $P_i|v\rangle = |i\rangle\langle i|v\rangle = |i\rangle v_i$

$= v_i |i\rangle$

random vector $|v\rangle$

Operators on a vector Space

An operator A is an action on a vector that generates another vector: $A: |v\rangle \longrightarrow |w\rangle$

$\hookrightarrow A|v\rangle = |w\rangle$

with $|v\rangle, |w\rangle \in \mathbb{V}$ (vector space)

<u>Linear Operators</u> $[\alpha, \beta \in \mathbb{C}]$: $A\left(\alpha|v\rangle + \beta|w\rangle\right) = \alpha A|v\rangle + \beta A|w\rangle$

<u>Anti-Linear Operators</u>: $A(\alpha|v\rangle + \beta|w\rangle) = \alpha^* A|v\rangle + \beta^* A|w\rangle$

Also $(A+B)|v\rangle = A|v\rangle + B|v\rangle$

$(AB)|v\rangle = A\left(B|v\rangle\right)$

<u>note</u>: generally $AB|v\rangle \neq BA|v\rangle$ order matters!!!

<u>Null operator</u>: $\mathbb{0}|v\rangle = |0\rangle \quad \forall |v\rangle \in \mathbb{V}$

Identity operator: $\mathbb{1}|v\rangle = |v\rangle \quad \forall |v\rangle \in V$

If $\{i\}$ is an orthonormal basis of V, then

$$\underbrace{\mathbb{1} = \sum_i |i\rangle\langle i|}_{\text{specific representation of } \mathbb{1}}$$

Operator inverse: A^{-1} defined as the operator that uniquely satisfies $A^{-1}A = AA^{-1} = \mathbb{1}$

Matrix Representation of an operator

Recall $|w\rangle = \mathbb{1}|w\rangle = \sum_i |i\rangle \underbrace{\langle i|w\rangle}_{w_i} = \sum_i w_i |i\rangle$

component / coefficient of the $|i\rangle$ basis vector

If $|w\rangle = A|v\rangle$, then we can write

$$\mathbb{1}|w\rangle = \mathbb{1} A \mathbb{1} |v\rangle$$

$$\Leftrightarrow \sum_k w_k |k\rangle = \left(\sum_i |i\rangle\langle i|\right) A \left(\sum_j |j\rangle\langle j|\right)|v\rangle$$

$$= \sum_{i,j} |i\rangle \underbrace{\langle i|A|j\rangle}_{} \underbrace{\langle j|v\rangle}_{v_j}$$

rename $\langle i|A|j\rangle = A_{ij} \in \mathbb{C}$

$$= \sum_{i,j} A_{ij} v_j |i\rangle$$

set $i=k$

$$\sum_i w_i |i\rangle = \sum_i \left[\sum_j A_{ij} v_j\right] |i\rangle$$

$$\Rightarrow w_i = \sum_j A_{ij} v_j$$

line column

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_n \end{pmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdots & A_{1n} \\ A_{21} & A_{22} & & & \vdots \\ A_{31} & & A_{33} & & \vdots \\ \vdots & & & \ddots & \vdots \\ A_{n1} & \cdots & \cdots & \cdots & A_{nn} \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{pmatrix}$$

Thus an operator A can be represented (in a specific basis) as a matrix

$$A \longrightarrow [A]_{ij} = \langle i | A | j \rangle = A_{ij}$$

line column

Alternate format: $$A = \sum_{i,j} |i\rangle\langle i|A|j\rangle\langle j|$$

$$= \sum_{i,j} A_{ij} \, |i\rangle\langle j|$$

$$\equiv \underbrace{A_{ij} \, |i\rangle\langle j|}_{\text{alternate notation}}$$

implied sum over repeated indices

Hermitian Operators

<u>Definition</u> The Adjoint A^{+} of an operator A is defined as the operator that satisfies the relation

$$\langle w | A v \rangle = \langle A^{+} w | v \rangle$$

<u>notation:</u> $A|v\rangle = |Av\rangle$

<u>Comment:</u> the "adjoint" A^{+} is sometimes referred to as the "Hermitian adjoint" or "Hermitian Conjugate"

If $A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & & \\ \vdots & & \ddots & \\ A_{n1} & & & A_{nn} \end{bmatrix}$

then $A^{+} = (A^{t})^{*}$ t = transpose

$$= \begin{bmatrix} A_{11}^{*} & A_{21}^{*} & \cdots & A_{n1}^{*} \\ A_{12}^{*} & A_{22}^{*} & & \\ \vdots & & \ddots & \\ A_{1n}^{*} & & & A_{nn}^{*} \end{bmatrix}$$

Definition: A is Hermitian if and only if $\boxed{A = A^{+}}$

In matrix notation, A is Hermitian iff $\boxed{A = (A^{t})^{*} \text{ or } A_{ij} = A_{ji}^{*}}$

Note 1: $\langle w | Av \rangle = \langle A^{+} w | v \rangle = \langle Aw | v \rangle$ if A is Hermitian

general: $\underbrace{A|v\rangle}_{\text{Ket format}} \longrightarrow \underbrace{\langle v | A^{+}}_{\text{Bra format}}$ A^{+} acts to the left

If A is Hermitian, then $\underbrace{A|v\rangle}_{\text{acts to right}} \longrightarrow \underbrace{\langle v | A}_{\text{acts to left}}$

Note 2: If A is Hermitian, then the notation $\langle v | A | w \rangle$ is well defined.

can act to the left can act to the right

Definition: Expectation value

The expectation value of an Operator A with the vector $|v\rangle$ is given by $\langle v|Av\rangle$.

If A is Hermitian, then the expectation value is $\langle v|A|v\rangle$ and is real, i.e. $\langle v|A|v\rangle \in \mathbb{R}$.

$A^\dagger = A$ if A is Hermitian

Proof: $\underbrace{\langle v|Av\rangle = \langle A^\dagger v|v\rangle}_{\text{Hermitian adjoint property}} = \underbrace{\langle v|Av\rangle^*}_{\text{definition of inner product}}$

$\Rightarrow \langle v|Av\rangle = \langle v|Av\rangle^* \Rightarrow \langle v|Av\rangle \in \mathbb{R}$

$\Leftrightarrow \langle v|A|v\rangle \in \mathbb{R}$

Stopped Here

Definition: Eigenvalues and eigenvectors

The vector $|\psi_\lambda\rangle$ is an eigenvector of an operator A (not necessarily Hermitian) with eigenvalue λ ($\lambda \in \mathbb{C}$) iff $\boxed{A|\psi_\lambda\rangle = \lambda|\psi_\lambda\rangle}$

↳ find λ by solving $\det[A - \lambda \mathbb{1}] = 0$ (determinant)

Spectral Theorem:

All the eigenvalues λ_i of a Hermitian operator A are real, i.e. $\lambda_i \in \mathbb{R}$, and the eigenvectors are all orthogonal and can be made form an eigenbasis $\{|\psi_{\lambda_i}\rangle\}$ that spans V

note: the set of eigenvalues $\{\lambda_i\}$ of a Hermitian operator A is often referred to as the spectrum of A.