Thursday, November 5, 2026

Spherical Harmonics (continued)

$\langle \vec{r} | \ell, m \rangle = Y_\ell^m(\theta, \phi)$ with $|\ell, m\rangle$ the eigenstates of $\vec{L}^2$ & L_z

i.e. $\vec{L}^2 |\ell, m\rangle = \hbar^2 \ell(\ell+1) |\ell, m\rangle$

$L_z |\ell, m\rangle = \hbar m |\ell, m\rangle$

A few additional properties of $Y_\ell^m(\theta, \phi)$ functions

More generally $\quad Y_\ell^{-m}(\theta, \phi) = (-1)^m Y_\ell^{*m}(\theta, \phi)$

for $m=0$: $Y_\ell^0(\theta, \phi) = \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos\theta)$

P_ℓ ← Legendre Polynomial

Orthonormality of Y_ℓ^m basis:

$$\int d\Omega \, Y_{\ell'}^{*m'}(\theta, \phi) Y_\ell^m(\theta, \phi) = \delta_{\ell\ell'} \delta_{mm'}$$

$$\Longleftrightarrow \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \, Y_{\ell'}^{*m'}(\theta, \phi) Y_\ell^m(\theta, \phi) = \delta_{\ell\ell'} \delta_{mm'}$$

Identity closure relation: The $Y_\ell^m(\theta,\phi)$ functions form a basis for any function $f(\theta,\phi)$

$0 \le \theta \le \pi$, $0 \le \phi < 2\pi$

$$\text{"}\mathbb{1}\text{"} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{+\ell} Y_\ell^{*m}(\theta',\phi') Y_\ell^m(\theta,\phi) = \delta(\cos\theta - \cos\theta')\,\delta(\varphi-\varphi')$$

$$= \frac{\delta(\theta-\theta')\,\delta(\varphi-\varphi')}{\sin\theta}$$

Euler Angles

The Euler angles provide a general description of the orientation of an object in 3D using 3 angles.

We consider two sets of 3D cartesian axes (orthonormal):

1) 3D space axes: x, y, z (fixed)
2) A set of axes attached to the object (x', y', z') which rotate with the object.
   - ↳ initially, the two axes are aligned: $\hat{x} = \hat{x}'$, $\hat{y} = \hat{y}'$, $\hat{z} = \hat{z}'$

The Euler angle prescription is to parametrize a general rotation (or orientation) for the object as follows:

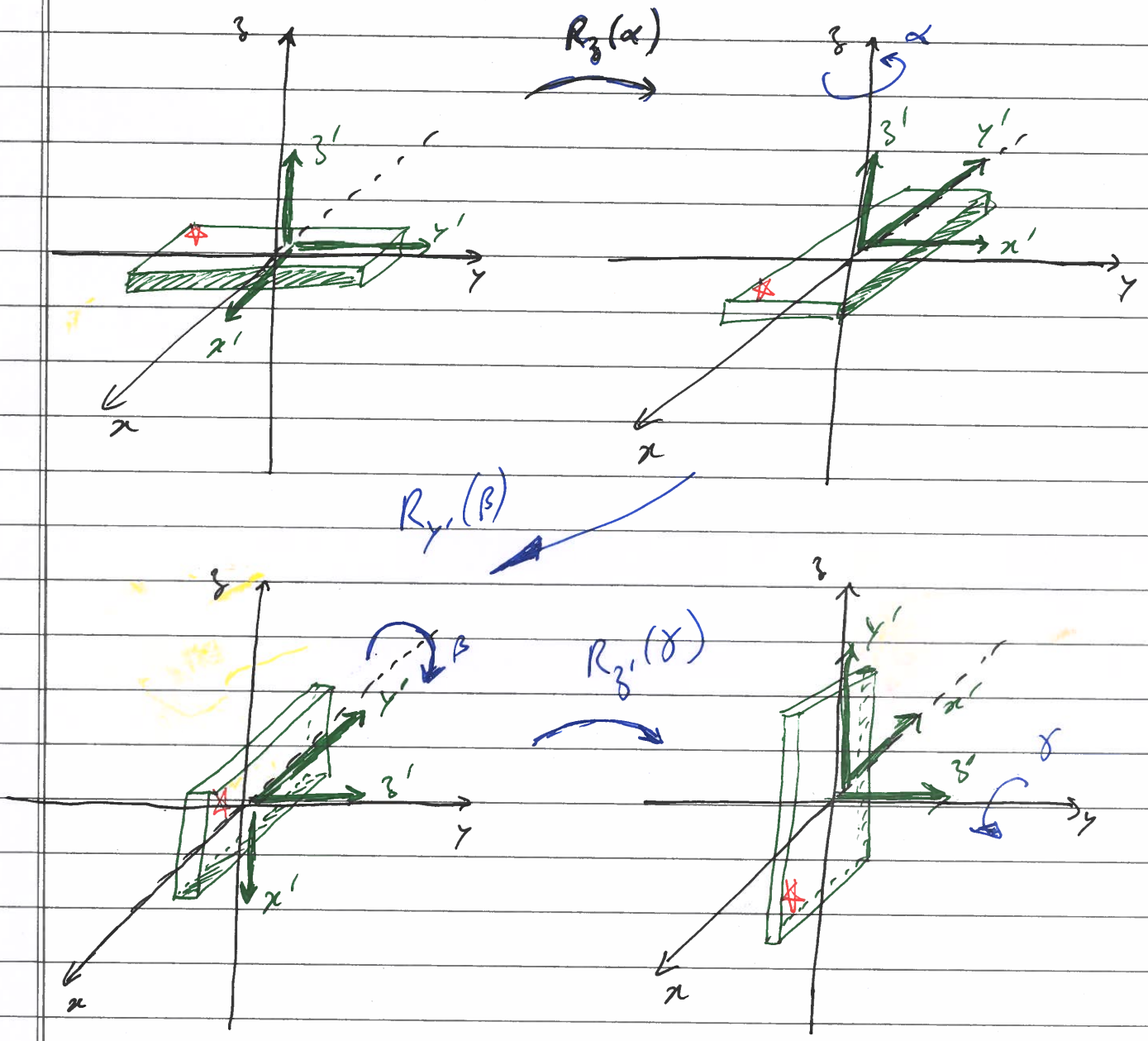
counterclockwise

$0 \le \alpha \le 2\pi$ 1) Rotate the object by α about the z-axis (i.e., z'-axis).

$0 \le \beta \le \pi$ 2) Rotate the object counterclockwise by β about the object's y'-axis.

$0 \le \gamma \le 2\pi$ 3) Rotate the object counterclockwise by γ about the object's z'-axis

Thus $\boxed{R(\alpha,\beta,\gamma) = R_{z'}(\gamma)\,R_{y'}(\beta)\,R_{z}(\alpha)}$

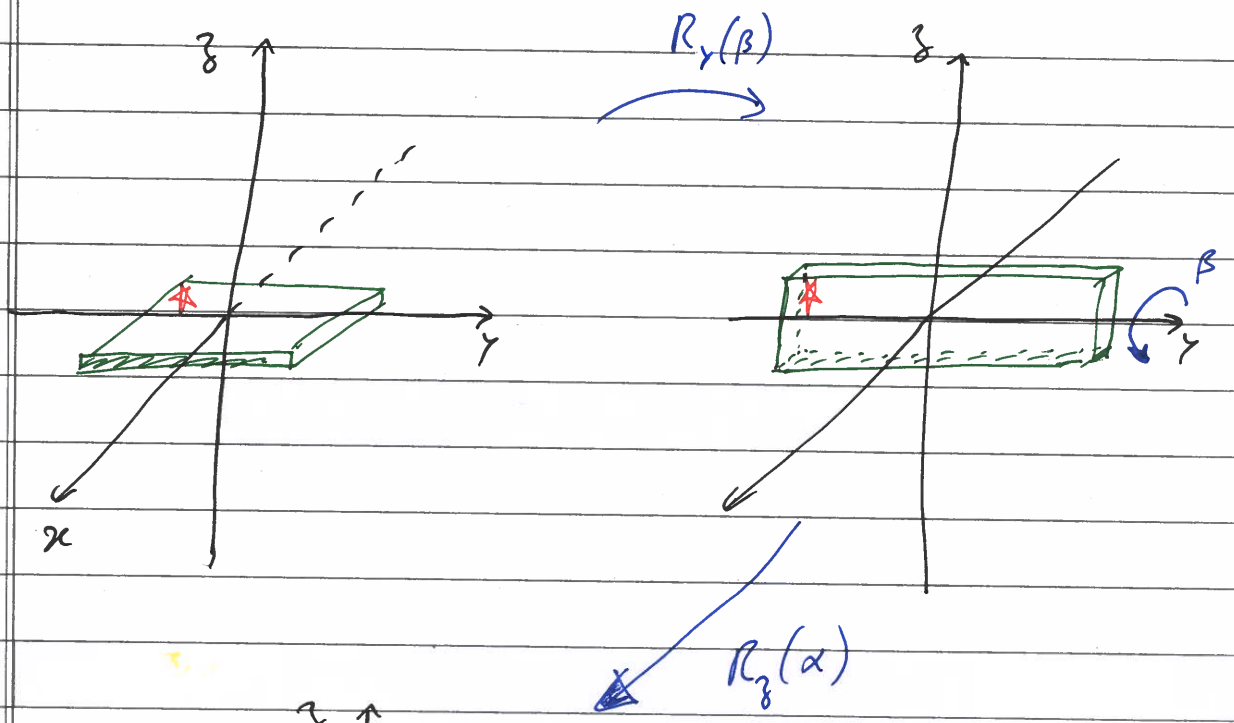


From a theoretical perspective, it is much more convenient to work entirely with rotations about the fixed 3D space axes, which remain unchanged as the object rotates.

↳ This can be achieved by using the following result

$$\boxed{R_{y'}(\beta)\,R_z(\alpha) = R_z(\alpha)\,R_y(\beta)}$$

Example: $\alpha = \pi/2$, $\beta = \pi/2$



$$\Rightarrow R_{y'}(\beta)\, R_z(\alpha) = R_z(\alpha) R_y(\beta)$$

Note: since $(AB)^{-1} = B^{-1}A^{-1}$ $\Rightarrow$

$$R_{y'}(\beta) = R_z(\alpha)\, R_y(\beta)\, R_z^{-1}(\alpha) \quad \text{E1}$$

$$R_{y'}^{-1}(\beta) = R_z(\alpha)\, R_y^{-1}(\beta)\, R_z^{-1}(\alpha) \quad \text{E2}$$

Similarly, we also have $R_{z'}(\gamma)\, R_{y'}(\beta) = R_{y'}(\beta)\, R_z(\gamma)$

$$\Rightarrow R_{z'}(\gamma) = R_{y'}(\beta)\, R_z(\gamma)\, R_{y'}^{-1}(\beta)$$

using E1 / using E2

$$= \left(R_z(\alpha) R_y(\beta) R_z^{-1}(\alpha)\right) R_z(\gamma) \left(R_z(\alpha) R_y^{-1}(\beta) R_z^{-1}(\alpha)\right)$$

$$= R_z(\gamma) \quad \text{(same axis rotations commute)}$$

$$\Rightarrow R_{z'}(\gamma) = R_z(\alpha)\, R_y(\beta)\, R_z(\gamma)\, R_y^{-1}(\beta)\, R_z^{-1}(\alpha) \quad \text{E3}$$

Returning to the original Euler expression:

$$R(\alpha,\beta,\gamma) = R_{z'}(\gamma)\,R_{y'}(\beta)\,R_z(\alpha)$$

using E1

using E3

$$= \left(R_z(\alpha)\,R_y(\beta)\,R_z(\gamma)\,R_y^{-1}(\beta)\,R_z^{-1}(\alpha)\right)\left(R_z(\alpha)\,R_y(\beta)\,R_z^{-1}(\alpha)\right)R_z(\alpha$$

$$= R_z(\alpha)\,R_y(\beta)\,R_z(\gamma)$$

thus $$R(\alpha,\beta,\gamma) = R_{z'}(\gamma)\,R_{y'}(\beta)\,R_z(\alpha) = R_z(\alpha)\,R_y(\beta)\,R_{z'}(\gamma)$$

Euler angles for fixed rotation axes

In QM, the Euler rotations then become

$$R(\alpha,\beta,\gamma) = R_z(\alpha)\,R_y(\beta)\,R_z(\gamma)$$

$$= e^{-i\frac{\alpha}{\hbar}J_z}\,e^{-i\frac{\beta}{\hbar}J_y}\,e^{-i\frac{\gamma}{\hbar}J_z}$$

Since $\left[\vec{J}^2, R(\alpha,\beta,\gamma)\right] = 0$, the rotation does not change j.

↳ m typically changes!

Also, we can write the $R(\alpha,\beta,\gamma)$ operator/transformation as

$$\langle j,m'|\,R(\alpha,\beta,\gamma)\,|j,m\rangle = e^{-im'\alpha}\,\langle j,m'|\,e^{-i\frac{\beta}{\hbar}J_y}\,|j,m\rangle\,e^{-im\gamma}$$

rotation matrix about y-axis

For example, in the case of spin-$\frac{1}{2}$ with $j=\frac{1}{2}$, we have

$$\langle j,m'| e^{-i\frac{\beta}{\hbar}J_y} |j,m\rangle = \cos\frac{\beta}{2} - i\sin\left(\frac{\beta}{2}\right)\sigma_y$$

$$= \begin{matrix} \\ \langle + | \\ \langle - | \end{matrix} \begin{matrix} \begin{matrix} |+\rangle & \quad |-\rangle \end{matrix} \\ \begin{bmatrix} \cos\beta/2 & -\sin\beta/2 \\ +\sin\beta/2 & \cos\beta/2 \end{bmatrix} \end{matrix}$$

3D QM systems

Free particle ($V=0$)

$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

Time independent Schrodinger equation: $-\frac{\hbar^2}{2m}\nabla^2\psi_E(\vec{r}) = E\psi_E(\vec{r})$

In cartesian coordinates, we get (using separation of variables)

Eigenstates: $\psi_{\vec{p}}(\vec{r}) = \frac{1}{\sqrt{2\pi\hbar}} e^{i\frac{p_x x}{\hbar}} \times \frac{1}{\sqrt{2\pi\hbar}} e^{i\frac{p_y y}{\hbar}} \times \frac{1}{\sqrt{2\pi\hbar}} e^{i\frac{p_z z}{\hbar}}$

$$= \frac{1}{(2\pi\hbar)^{3/2}} e^{i\frac{\vec{p}\cdot\vec{r}}{\hbar}} = \frac{1}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{r}}$$

defining: $\vec{p} = \hbar\vec{k}$

Note: $\int d^3\vec{r}\ \psi^*_{\vec{p}'}(\vec{r})\,\psi_{\vec{p}}(\vec{r}) = \delta^{(3)}(\vec{p}-\vec{p}')$

$$\int d^3\vec{p}\ \psi^*_{\vec{p}}(\vec{r}')\,\psi_{\vec{p}}(\vec{r}) = \delta^{(3)}(\vec{r}-\vec{r}')$$

Eigenenergy: $E_{\vec{p}} = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) = \frac{\vec{p}^2}{2m} = \frac{\hbar^2\vec{k}^2}{2m}$