Thursday, September 10, 2026

Postulates of Quantum Mechanics [Cohen-Tannoudji]

Postulate #1: System is defined by a ket $|\psi(t)\rangle$ on a Hilbert Space V.

Postulate #2: Observables are given by Hermitian operators acting in V.

Postulate #3: A measurement for an observable A can only yield an eigenvalue of A. [eigenbasis of A spans V]

Postulate #4a (discrete, non-degenerate eigenvalue)

When the observable A is measured on a system in the normalized state $|\psi\rangle$ (i.e. $\langle\psi|\psi\rangle = 1$), then the probability $P(\lambda_n)$ of obtaining the eigenvalue λ_n of A is

$$P(\lambda_n) = |\langle\lambda_n|\psi\rangle|^2$$

where $|\lambda_n\rangle$ is the normalized eigenvector for λ_n (of A).

Note: $|\psi\rangle = \sum_m c_m |\lambda_m\rangle \Rightarrow P(\lambda_n) = |\langle\lambda_n|\psi\rangle|^2$

$$= |\langle\lambda_n|\sum_m c_m|\lambda_m\rangle|^2$$

$$= \Big|\sum_m c_m \underbrace{\langle\lambda_n|\lambda_m\rangle}_{\delta_{nm}}\Big|^2 = |c|^2$$

Postulate 4b (discrete, degenerate eigenvalue)

If the degeneracy of λ_n is g_n ($g_n = 2, 3, \ldots$), then $P(\lambda_n)$ is

$$P(\lambda_n) = \sum_{i=1}^{g_n} |\langle\lambda_{n_i}|\psi\rangle|^2 = \sum_{i=1}^{g_n} |c_{n_i}|^2$$

where $\{|\lambda_{n_i}\rangle\}$ form an orthonormal eigenbasis for all eigenvectors with eigenvalue λ_n.

subspace dimension $= g_n$

<u>Postulate 4c</u> (continuous spectrum)

If A has a continuous spectrum of eigenvalues $\{\alpha\}$, then the probability to get a measurement value between α and $\alpha + d\alpha$ is given by

$$dP(\alpha) = |\langle v_\alpha | \psi \rangle|^2 \, d\alpha$$

where $|v_\alpha\rangle$ is the eigenvector corresponding to eigenvalue α.

Expectation value of A for $|\psi\rangle$ = Average value of A using $|\psi\rangle$ for many measurements

$$\langle A \rangle_\psi = \, ? \, = \text{average of } A \text{ for } |\psi\rangle$$

$$P_{exp}(\lambda_n) = \frac{\text{Number of times you get } \lambda_n}{\text{Number of measurements}} = \left.\frac{N(\lambda_n)}{N_{total}}\right|_{N_{total} \to +\infty}$$

$$= \text{Experimental definition} = |\langle \lambda_n | \psi \rangle|^2$$

[Postulate 4a]

$$\langle A \rangle_\psi = \sum_n \lambda_n \underbrace{P_{exp}(\lambda_n)}_{|\langle \lambda_n | \psi \rangle|^2}$$

$$= \sum_n \lambda_n |\langle \lambda_n | \psi \rangle|^2$$

$\lambda_n |\lambda_n\rangle = A|\lambda_n\rangle$

$$= \sum_n \lambda_n \underbrace{\langle \lambda_n | \psi \rangle^*}_{\langle \psi | \lambda_n \rangle} \langle \lambda_n | \psi \rangle = \sum_n \lambda_n \langle \psi | \lambda_n \rangle \langle \lambda_n | \psi \rangle$$

$$= \sum_n \langle \psi | A | \lambda_n \rangle \langle \lambda_n | \psi \rangle = \langle \psi | A \Big(\underbrace{\sum_n |\lambda_n\rangle\langle\lambda_n|}_{= \mathbb{1}} \Big) |\psi\rangle$$

$$= \langle \psi | A | \psi \rangle$$

$\Rightarrow$ Average value for A for $|\psi\rangle$ (over many measurements)

$$= \langle A \rangle_\psi = \underbrace{\langle \psi | A | \psi \rangle}_{\text{expectation value of } A}$$

Postulate #5: If the measurement of the physical quantity $\mathcal{A}$ on the system in the state $|\psi\rangle$ gives the eigenvalue λ_n, then the state of the system immediately after the measurement $|\psi'\rangle$ is $|\psi'\rangle = |\lambda_n\rangle$

If λ_n is degenerate, then $|\psi'\rangle = \underbrace{\dfrac{P_n |\psi\rangle}{\sqrt{\langle \psi | P_n | \psi \rangle}}}_{\text{Normalized projection}}$

P_n = Projection operator onto the λ_n subspace

Note 1: The state $|\psi\rangle$ collapses to state $\underbrace{|\psi'\rangle}_{\text{eigenstate of } A}$.

Note 2: Could the measurement postulate be derived from the other QM postulates?

Projection operator $P_n = |\lambda_n\rangle\langle\lambda_n|$ = project onto $|\lambda_n\rangle$

$$\equiv \sum_{i=1}^{g_n} |\lambda_{n_i}\rangle\langle\lambda_{n_i}| = \text{project onto degenerate } \{|\lambda_{n_i}\rangle\} \text{ subspace}$$

Embedded in these postulates is an important counter-intuitive non-classical aspect of Quantum Mechanics:

$$\boxed{\text{Specific Values cannot be ascribed to observables prior to measurement; such values are only the outcome of measurements.}}$$

Prof. Jo Dudek

Postulate #6: The time evolution of the state $|\psi(t)\rangle$ is given by the Schrödinger equation:

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H(t)\, |\psi(t)\rangle$$

where $H(t)$ is the Hamiltonian observable/operator associated with the total energy of the system.

Note: The time evolution of a wavefunction/ket $|\psi(t)\rangle$ is completely deterministic until a measurement is made.

[decoherence = measurements by the environment]

Non-Commuting Observables

We consider two "incompatible" observables A & B, i.e. they do not commute: $AB \neq BA$ or $[A,B] = AB - BA \neq 0$

We have two eigenbases: $A|a_i\rangle = a_i |a_i\rangle$

$B|b_j\rangle = b_j |b_j\rangle$

and no simultaneous eigenkets: $|a_i\rangle \neq |b_j\rangle$ for all i,j

Q: What is the probability that ~~x~~ we will get a_i under an A measurement and then get b_j under a subsequent B measurement?

$$P_\psi(a_i \to b_j) = P_{|a_i\rangle}(b_j) \cdot P_\psi(a_i) = |\langle b_j | a_i \rangle|^2 \, |\langle a_i | \psi \rangle|^2$$

$|\psi\rangle$ = initial state

We can sum over all the intermediate possibilities, i.e. a_i

$$\Rightarrow \sum_{a_i} |\langle b_j | a_i \rangle|^2 \, |\langle a_i | \psi \rangle|^2 = P_{\text{"classical"}, |\psi\rangle}(b_j)$$

= Probability to get b_j with no knowledge of a_i result.

If we don't do the intermediate ~~B~~ A measurement, then according to QM we get

$$P_{\text{"quantum"}} = P_{|\psi\rangle}(b_j) = |\langle b_j | \psi \rangle|^2 = \left| \sum_{a_i} \langle b_j | a_i \rangle \langle a_i | \psi \rangle \right|^2$$

insert $\mathbb{1}$

$$P_{|\psi\rangle}(b_j) \neq P_{\text{"classical"}, |\psi\rangle}(b_j)$$

$|\psi\rangle$ collapses to $|a_i\rangle$

$\Rightarrow$ the intermediate measurement of A has an effect on the final outcome of the B measurement, even if we sum over all the intermediate outcomes (a_i)

Q: What is the spread or variance on measurements of an observable?

We define the variance of ~~a~~ measurements of A on $|\psi\rangle$ as

$$\boxed{\begin{aligned}\Delta A_\psi^2 &= \langle\psi|(A - \langle A\rangle_\psi \mathbb{1})^2|\psi\rangle \\ &= \langle\psi|A^2|\psi\rangle - (\langle\psi|A|\psi\rangle)^2 \\ &= \langle\psi|A^2|\psi\rangle - \langle A\rangle_\psi^2\end{aligned}}$$

Next, Consider two observable that do not commute $[A,B] \neq 0$

$\Delta A = A - \langle A\rangle_\psi \mathbb{1}$ and $\Delta A|\psi\rangle = |\alpha\rangle \Rightarrow \langle\psi|(\Delta A)^2|\psi\rangle$

$$= \langle\psi|\Delta A\,\Delta A|\psi\rangle$$
$$= \langle\alpha|\alpha\rangle$$

$\Delta B = B - \langle B\rangle_\psi \mathbb{1}$ and $\Delta B|\psi\rangle = |\beta\rangle \Rightarrow \langle\psi|(\Delta B)^2|\psi\rangle$

$$= \langle\psi|\Delta B\,\Delta B|\psi\rangle = \langle\beta|\beta\rangle$$

Next, we look at the product of the variances:

$$\langle\psi|(\Delta A)^2|\psi\rangle \cdot \langle\psi|(\Delta B)^2|\psi\rangle = \langle\alpha|\alpha\rangle \cdot \langle\beta|\beta\rangle$$

The schwarz inequality states that $\boxed{\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2}$

Furthermore, $\langle\alpha|\beta\rangle = \langle\psi|\Delta A\,\Delta B|\psi\rangle$

We can write $\Delta A \cdot \Delta B = \frac{1}{2}\underbrace{[\Delta A, \Delta B]}_{=[A,B]} + \frac{1}{2}\{\Delta A, \Delta B\}$

$$= \frac{1}{2}[A,B] + \frac{1}{2}\{\Delta A, \Delta B\}$$

Recall: $(AB)^\dagger = B^\dagger A^\dagger$

A, B are Hermitian

Thus $[A,B]^\dagger = (AB-BA)^\dagger = B^\dagger A^\dagger - A^\dagger B^\dagger \xrightarrow{} BA - AB = [B,A]$
$= -[A,B]$

$\Rightarrow [A,B]^\dagger = -[A,B] \Rightarrow \boxed{[A,B] \text{ is antiHermitian}}$

Similarly
$$\{\Delta A, \Delta B\}^\dagger = (\Delta A \Delta B + \Delta B \Delta A)^\dagger = \Delta B^\dagger \Delta A^\dagger + \Delta A^\dagger \Delta B^\dagger$$
$$= \Delta B \Delta A + \Delta A \Delta B = \{\Delta A, \Delta B\}$$

$\Rightarrow \{\Delta A, \Delta B\}^\dagger = +\{\Delta A, \Delta B\} \Rightarrow \boxed{\{\Delta A, \Delta B\} \text{ is Hermitian}}$

Thus,
$$\langle\psi|\Delta A \Delta B|\psi\rangle = \frac{1}{2}\underbrace{\langle\psi|[A,B]|\psi\rangle}_{\substack{\text{imaginary} \\ = Im}} + \frac{1}{2}\underbrace{\langle\psi|\{\Delta A, \Delta B\}|\psi\rangle}_{\substack{\text{real} \\ = Re}}$$

$$= Re + i\, Im$$

Back to Schwarz inequality

$$\langle\psi|(\Delta A)^2|\psi\rangle\langle\psi|(\Delta B)^2|\psi\rangle \geq \underbrace{\left|\frac{1}{2}(Re + i\, Im)\right|^2}_{\frac{1}{4}(Re^2 + Im^2) \geq \frac{1}{4} Im^2}$$

$$\Rightarrow \boxed{\langle\psi|(\Delta A)^2|\psi\rangle\langle\psi|(\Delta B)^2|\psi\rangle \geq \frac{1}{4}\left|\langle\psi|[A,B]|\psi\rangle\right|^2}$$

Generalized Heisenberg Uncertainty ~~Relation~~ Inequality

$$\Longleftrightarrow \Delta A_\psi \cdot \Delta B_\psi \geq \frac{1}{2}\left|\langle [A,B] \rangle_\psi\right|$$