Tuesday, September 8, 2026

recall: $\langle w | A v \rangle = \langle A^{\dagger} w | v \rangle$

Recall: A is Hermitian iff $A = A^{\dagger}$

$\iff A = (A^t)^* \iff A_{ij} = A_{ji}^*$

Definition: A linear operator B is anti-Hermitian iff $A^{\dagger} = -A$

$\iff -A = (A^t)^* \iff A_{ij} = -A_{ji}^*$

Theorem: Every linear operator M can be written as a sum of a Hermitian and anti-Hermitian operator:

$$M = \underbrace{\frac{M + M^{\dagger}}{2}}_{\text{Hermitian}} + \underbrace{\frac{M - M^{\dagger}}{2}}_{\text{anti-Hermitian}}$$

Definition: Eigenvalues and eigenvectors

The vector $|\psi_\lambda\rangle$ is an eigenvector of an $\underbrace{\text{operator } A}_{\text{not necessarily Hermitian}}$ with $\underbrace{\text{eigenvalue } \lambda}_{\lambda \in \mathbb{C}}$ iff $\boxed{A|\psi_\lambda\rangle = \lambda|\psi_\lambda\rangle}$

↳ find λ by solving $\det[A - \lambda \mathbb{1}] = 0$

↑ determinant

Spectral Theorem:

All the eigenvalues λ_i of a Hermitian operator A are real, i.e. $\lambda_i \in \mathbb{R}$, and the eigenvectors are all orthogonal and can be made to form an orthonormal eigenbasis $\{|\psi_{\lambda_i}\rangle\}$ that spans V

$\boxed{\text{In this eigenbasis, } A \text{ is diagonal with form } A = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 \ddots & \\ 0 & & \lambda_n \end{pmatrix}}$

note: the set of eigenvalues $\{\lambda_i\}$ of a Hermitian operator A is often referred to as the spectrum of A.

Unitary operators

A linear operator U is unitary iff $\underbrace{U^\dagger = U^{-1}}_{\text{adjoint } = \text{ inverse}}$

i.e. $U^\dagger U = U U^\dagger = \mathbb{1}$

<u>Important:</u> Unitary operators are those that preserve the inner product on a complex vector space:

$$|v'\rangle = U|v\rangle = |Uv\rangle$$
$$|w'\rangle = U|w\rangle = |Uw\rangle$$

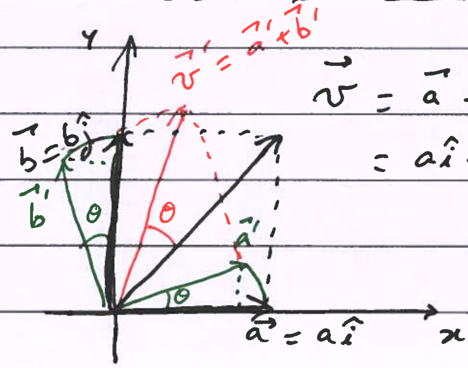
$$\langle v'|w'\rangle = \langle Uv|Uw\rangle = \langle v'|Uw\rangle$$
$$= \langle U^\dagger v'|w\rangle \quad \text{adjoint definition}$$
$$= \langle U^\dagger U v|w\rangle \quad U^\dagger U = \mathbb{1}$$
$$= \langle v|w\rangle$$

$\Rightarrow \langle Uv|Uw\rangle = \langle v|w\rangle$ inner product is unchanged when applying a unitary operator to both vectors.

<u>Note:</u> If U is real, then $U^\dagger = U^t = [U]_{ji}$ (if $U = [U]_{ij}$)
$= U^{-1}$

<u>Examples:</u> rotations, parity/mirror flip

<u>Rotation in 2D example:</u> Matrix to rotate a vector by angle θ



$\vec{v} = \vec{a} + \vec{b}$
$= a\hat{i} + b\hat{j}$

$$\vec{a} = \begin{pmatrix} a \\ 0 \end{pmatrix} \xrightarrow{R(\theta)} \vec{a}' = \begin{pmatrix} a\cos\theta \\ a\sin\theta \end{pmatrix}$$

$$\vec{b} = \begin{pmatrix} 0 \\ b \end{pmatrix} \xrightarrow{R(\theta)} \vec{b}' = \begin{pmatrix} -b\sin\theta \\ b\cos\theta \end{pmatrix}$$

by inspection: $R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = R_{ij}(\theta)$

$R(\theta)$ is unitary, since it preserves norm and the dot product

Note: $R^{-1}(\theta) = R(-\theta) = \underbrace{R_{ji}}_{\text{transpose}}(\theta)$

Change of Basis [see Sakurai, section 1.5]

Consider an orthonormal basis $\{|i\rangle\}$ and a different orthonormal basis $\{|i'\rangle\}$.

Objective: Relate the $\{|i\rangle\}$ and $\{|i'\rangle\}$ vectors

Suppose that the linear transformation M relates $\{|i'\rangle\}$ and $\{|i\rangle\}$ such that

$$|i'\rangle = \sum_j M_{ji'} |j\rangle$$

If $|k\rangle \in \{|i\rangle\}$, then

$$\langle k|i'\rangle = \sum_j M_{ji'} \underbrace{\langle k|j\rangle}_{\delta_{kj} \text{ (orthonormality)}} = M_{ki'}$$

Thus we can obtain the elements of the change of basis matrix $M_{ki'} = \langle k|i'\rangle$

Consider the inverse transformation N such that

$$|i\rangle = \sum_{j'} N_{j'i} |j'\rangle$$

if $|k'\rangle \in \{|i'\rangle\}$, then

$$\langle k'|i\rangle = \sum_{j'} N_{j'i} \underbrace{\langle k'|j'\rangle}_{\delta_{k'j'}} \quad \text{(orthonormality)}$$

$$= N_{j'i}$$

However $\underbrace{\langle k'|i\rangle = \langle i|k'\rangle^*}_{\text{definition of inner product}} = (M_{ik'})^*$ adjoint property

$$= M^+_{k'i}$$

$$\Rightarrow N_{j'i} = M^+_{k'i} \quad \Rightarrow \quad N = M^+$$

thus $\{|i\rangle\} \xrightarrow{M} \{|i'\rangle\} \xrightarrow{N=M^+} \{|i\rangle\}$

$$\underbrace{M^+ M}|i\rangle = |i\rangle$$

$$M^+ M = \mathbb{1} \Rightarrow M^+ = M^{-1}$$

$\Rightarrow M$ is unitary

Note: $\det(\underbrace{[U][U^+]}_{=\mathbb{1} \text{ for } U = \text{unitary}}) = \det(U)\det(U^+) = \det(\mathbb{1}) = 1$

$$= \det(U)\det((U^t)^*)$$

$$= \det(U)[\underbrace{\det(U^t)}_{=\det U}]^*$$

$$= |\det(U)|^2 \Rightarrow \boxed{|\det(U)| = 1}$$

U = unitary

Example: spin-1/2 system

$$|S_z,+\rangle = |+\rangle = |\uparrow\rangle \quad \text{and} \quad |S_z,-\rangle = |-\rangle = |\downarrow\rangle$$

$$|S_x,+\rangle = \frac{1}{\sqrt{2}}\Big[|+\rangle + |-\rangle\Big] \quad \text{and} \quad |S_x,-\rangle = \frac{1}{\sqrt{2}}\Big[|+\rangle - |-\rangle\Big]$$

Also $$|S_y,\pm\rangle = \frac{1}{\sqrt{2}}\Big[|+\rangle \underbrace{\pm i}_{\text{important}}|-\rangle\Big]$$

$$\{|i'\rangle\} = \{|S_x \pm\rangle\} \quad \text{and} \quad \{|i\rangle\} = \{|\pm\rangle\}$$

M is thus $$M = \begin{bmatrix} \langle + | S_x,+\rangle & \langle + | S_x,-\rangle \\ \langle - | S_x,+\rangle & \langle - | S_x,-\rangle \end{bmatrix}$$

$$= \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

N is thus $$N = \begin{bmatrix} \langle S_x,+ | +\rangle & \langle S_x,+ | -\rangle \\ \langle S_x,- | +\rangle & \langle S_x,- | -\rangle \end{bmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Note: $N = M^\dagger = M^{-1} \underset{\text{in this case}}{=} M$

$$M M^{-1} = M^2 = \mathbb{1}$$

Postulates of Quantum Mechanics (Cohen-Tannoudji)

Postulate #1: At a fixed time t_0, the state of a physical system is defined by specifying a ket $|\psi(t_0)\rangle$ belonging to a Hilbert space $\mathbb{V}$.

quantum superposition
quantum entanglement

Postulate #2: Every measurable physical quantity $\mathcal{A}$ is described by a Hermitian operator A acting in $\mathbb{V}$. This operator is an observable.

Note: Observables do not necessarily commute : $BA|\psi\rangle \neq AB|\psi\rangle$

Postulate #3: The only possible result of a measurement of a physical quantity $\mathcal{A}$ is one of the eigenvalues of the corresponding observable A.

Note: Since A is Hermitian, its eigenbasis $\{|\lambda_n\rangle\}$ spans $\mathbb{V}$, so any state $|\psi\rangle$ can be written as a linear combination of $|\lambda_n\rangle$

$$\Rightarrow \quad |\psi\rangle = \sum_n c_n |\lambda_n\rangle$$

Postulate #4: When the physical quantity $\mathcal{A}$ is measured on a system in the normalized state $|\psi\rangle$ (i.e., $\langle\psi|\psi\rangle = 1$), the probability $P(\lambda_n)$ of obtaining the eigenvalue λ_n of the observable A is

$$P(\lambda_n) = |\langle\lambda_n|\psi\rangle|^2$$

where $|\lambda_n\rangle$ is the normalized eigenvector for λ_n (of A).

Note: If λ_n is degenerate or continuous, then $P(\lambda_n)$ is modified somewhat.

Postulate #5: If the measurement of the physical quantity $\mathcal{A}$ on the system in the state $|\psi\rangle$ gives the eigenvalue λ_n, then the state of the system immediately after the measurement $|\psi'\rangle$ is $|\psi'\rangle = |\lambda_n\rangle$

If λ_n is degenerate, then $$|\psi'\rangle = \underbrace{\frac{P_n|\psi\rangle}{\sqrt{\langle\psi|P_n|\psi\rangle}}}_{\text{Normalized projection}}$$

P_n = Projection operator onto the λ_n subspace

Note 1: The state $|\psi\rangle$ collapses to state $|\psi'\rangle$.

Note 2: Could the measurement postulate be derived from the other QM postulates?

Postulate #6: The time evolution of the state $|\psi(t)\rangle$ is given by the Schrödinger equation:

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H(t)|\psi(t)\rangle$$

where $H(t)$ is the Hamiltonian observable/operator associated with the total energy of the system.