Tuesday, September 15, 2026

Last time, we started looking at the product of variances (i.e. work towards a generalized uncertainty relation) for two operators (A & B) for some state (ψ)

$$\langle\psi|(\Delta A)^2|\psi\rangle \cdot \langle\psi|(\Delta B)^2|\psi\rangle = \langle\alpha|\alpha\rangle \cdot \langle\beta|\beta\rangle$$

for $\Delta A|\psi\rangle = |\alpha\rangle$

$\Delta B|\psi\rangle = |\beta\rangle$

Schwarz inequality guarantees: $\boxed{\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2}$

furthermore $\langle\alpha|\beta\rangle = \langle\psi|\Delta A\,\Delta B|\psi\rangle$

we can write $\Delta A \cdot \Delta B = \frac{1}{2}\underbrace{[\Delta A, \Delta B]}_{\Delta A\Delta B - \Delta B\Delta A \,=\, [A,B]} + \frac{1}{2}\underbrace{\{\Delta A, \Delta B\}}_{\Delta A\Delta B + \Delta B\Delta A}$

recall: $\Delta A = A - \langle A\rangle_\psi \mathbb{1}$

$$= \frac{1}{2}[A,B] + \frac{1}{2}\{\Delta A, \Delta B\}$$

Recall: $(AB)^\dagger = B^\dagger A^\dagger$

quick proof: $\langle w|(AB)v\rangle = \langle (AB)^\dagger w|v\rangle$

$= \langle A^\dagger w|Bv\rangle$

$= \langle B^\dagger A^\dagger w|v\rangle$

$B^\dagger A^\dagger = (AB)^\dagger$

$$\text{thus}\quad [A,B]^\dagger = (AB-BA)^\dagger = B^\dagger A^\dagger - A^\dagger B^\dagger \overset{A,B \text{ are Hermitian}}{=} BA - AB = [B,A]$$
$$= -[A,B]$$

$$\Rightarrow [A,B]^\dagger = -[A,B] \Rightarrow \boxed{[A,B] \text{ is antiHermitian}}$$

similarly

$$\{\Delta A, \Delta B\}^\dagger = (\Delta A \Delta B + \Delta B \Delta A)^\dagger = \Delta B^\dagger \Delta A^\dagger + \Delta A^\dagger \Delta B^\dagger$$
$$= \Delta B \Delta A + \Delta A \Delta B = \{\Delta A, \Delta B\}$$

$$\Rightarrow \{\Delta A, \Delta B\}^\dagger = +\{\Delta A, \Delta B\} \Rightarrow \boxed{\{\Delta A, \Delta B\} \text{ is Hermitian}}$$

Thus,

$$\langle \psi | \Delta A \Delta B | \psi \rangle = \frac{1}{2} \underbrace{\langle \psi | [A,B] | \psi \rangle}_{\substack{\text{imaginary} \\ = Im}} + \frac{1}{2} \underbrace{\langle \psi | \{\Delta A, \Delta B\} | \psi \rangle}_{\substack{\text{real} \\ = Re}}$$

$$= Re + i\, Im$$

Back to Schwarz inequality

$$\langle \psi | (\Delta A)^2 | \psi \rangle \langle \psi | (\Delta B)^2 | \psi \rangle \geq \underbrace{\left| \frac{1}{2} (Re + i\, Im) \right|^2}_{\frac{1}{4}\left(Re^2 + Im^2\right) \geq \frac{1}{4} Im^2}$$

$$\Rightarrow \boxed{\langle \psi | (\Delta A)^2 | \psi \rangle \langle \psi | (\Delta B)^2 | \psi \rangle \geq \frac{1}{4} \left| \langle \psi | [A,B] | \psi \rangle \right|^2}$$

Generalized Heisenberg Uncertainty ~~Rela~~ Inequality

$$\Longleftrightarrow \Delta A_\psi \cdot \Delta B_\psi \geq \frac{1}{2} \left| \langle [A,B] \rangle_\psi \right|$$

Position Operator & eigenkets

We postulate the existence of the position operator X and its eigenstate $|x\rangle$: $X|x\rangle = x|x\rangle$

particle at position "x"

x has units of length (meters)

Q: What is $\langle x|x'\rangle$?

$$\langle x|x'\rangle = \begin{cases} 0 & \text{for } x \neq x' \\ \text{"}1\text{"} & \text{for } x = x' \end{cases}$$

if $\{|x\rangle\}$ is an "orthonormal" basis

$$= \delta(x-x')$$

Dirac delta function

Completeness & identity operator: $\sum_i |i\rangle\langle i| \rightarrow \int dx\, |x\rangle\langle x|$

$$= \mathbb{1}$$

Projection of a state $|\psi\rangle$ onto the $\{|x\rangle\}$ basis

$$|\psi\rangle = \mathbb{1}|\psi\rangle = \sum_i |i\rangle\langle i|\psi\rangle = \sum_i \underbrace{\langle i|\psi\rangle}_{\psi_i}|i\rangle = \sum_i \psi_i|i\rangle$$

$$\hookrightarrow |\psi\rangle = \mathbb{1}|\psi\rangle = \int dx\, |x\rangle\langle x|\psi\rangle = \int dx\, \underbrace{\langle x|\psi\rangle}_{\psi_x \to \psi(x)}|x\rangle$$

$$= \int dx\, \psi(x)|x\rangle$$

Q: What is the probability to measure a particle in state $|\psi\rangle$ at position x?

→ A: (postulate 4) $dP(x) = |\underbrace{\langle x|\psi\rangle}|^2 dx$ (postulate 4c)

λ_x $\langle\lambda_x|$

probability distribution density: $P(x) = |\psi(x)|^2$

$\Rightarrow \quad dP(x) = |\psi(x)|^2 dx$

For consistency, we require $\boxed{\int_{\text{all } x} dP(x) = \int_{\text{all } x} |\psi(x)|^2 dx = 1}$

Translation operator and Momentum Operator

Objective: Motivate the commutation relation $[X, P] = i\hbar$.

The translation operator $T(dx)$ takes a state localized at x (i.e., $|x\rangle$) and translates it to $x + dx$:

$$T(dx)|x\rangle = |x + dx\rangle$$

We require that the T operator:

1) Conserve probability: $|\psi'\rangle = T|\psi\rangle \Rightarrow \langle\psi'|\psi'\rangle = 1$ ($T(dx)$)

$$\Leftrightarrow \langle\psi|T^\dagger T|\psi\rangle = 1$$

$$\Rightarrow T^\dagger T = 1 \Rightarrow T \text{ is unitary}$$

$$\Rightarrow T^\dagger = T^{-1}$$

2) Translating by dx and then dx' should be the same as translating by $dx + dx'$:

$$T(dx')T(dx) = T(dx + dx')$$

3) Translating by dx and then $-dx$ should leave the system unchanged:

$$T(-dx)\,T(dx) = \mathbb{1} \quad \Rightarrow \quad T(-dx) = T^{-1}(dx)$$

4) If $dx \to 0$, then there is no translation

$$\lim_{dx \to 0} T(dx) = \mathbb{1}$$

A candidate operator (satisfying these requirements) is

$$T(dx) = \mathbb{1} - i \frac{dx}{\hbar} P$$

Hermitian operator with unit of momentum

constant with unit $[E][t] \equiv [J][s]$

2) $T(dx')T(dx) = T(dx+dx') +$ "$dx\,dx'$" term

1) & 3) $T(-dx)\,T(dx) = T^{\dagger}(dx)\,T(dx) = \mathbb{1} -$ "dx^2" term

4) $\lim_{dx \to 0} T(dx) = \mathbb{1}$

Order of Operations : Commutator

i) Translate then measure position

$$X\,T(dx)|x\rangle = X|x+dx\rangle = (x+dx)|x+dx\rangle$$

ii) Measure position then translate

$$T(dx)\,X|x\rangle = T(dx)\,x|x\rangle = x|x+dx\rangle$$

x = scalar

$$\Rightarrow [X, T(dx)]|x\rangle = (X\,T(dx) - T(dx)\,X)|x\rangle$$

$$= [(x+dx) - x]|x+dx\rangle = dx\,\underbrace{|x+dx\rangle}_{T(dx)|x\rangle}$$

$$= dx\,T(dx)|x\rangle$$

$$= dx\left(\mathbb{1} - i\frac{dx}{\hbar}P\right)|x\rangle$$

$$= dx|x\rangle - \underbrace{i\frac{(dx)^2}{\hbar}}_{\to 0 \text{ for } dx \text{ small}} P|x\rangle$$

So for $\underline{dx}$ sufficiently small, we have $[X, T(dx)] = dx\,\mathbb{1}$

$$\Rightarrow \quad [X, \underbrace{T(dx)}_{\mathbb{1} - i\frac{dx}{\hbar}P}] = dx \Leftrightarrow \underbrace{[X, \mathbb{1}]}_{=0} - i\frac{dx}{\hbar}[X, P] = dx$$

$$\Rightarrow \boxed{[X, P] = i\hbar\,\mathbb{1}}$$

Finite size translation Δx:

$$|x + \Delta x\rangle = T(\Delta x)|x\rangle = \lim_{N\to\infty}\left(1 - \frac{i}{\hbar}\underbrace{\left(\frac{\Delta x}{N}\right)}_{dx \text{ for } N \to \text{ large}}P\right)^N |x\rangle$$

$$= e^{-i\frac{\Delta x}{\hbar}P}|x\rangle$$

$$\Rightarrow \boxed{T(\Delta x) = e^{-i\frac{\Delta x}{\hbar}P}}$$

$\Rightarrow$ $\boxed{\text{P (i.e. momentum) is a generator of translations.}}$

Heisenberg uncertainty relation

$$\langle\psi|\Delta X^2|\psi\rangle\langle\psi|\underbrace{\Delta P^2}_{P|\psi\rangle = ?}|\psi\rangle \geq \frac{1}{4}\left|\langle\psi|\underbrace{[X,P]}_{i\hbar}|\psi\rangle\right|^2$$

$$\Rightarrow \boxed{\Delta X\,\Delta P \geq \frac{\hbar}{2}}$$

Since $[X, P] = i\hbar$, states/kets $\underline{\text{cannot}}$ be simultaneous eigenstates of X and P.

Q: How do we calculate $P|\psi\rangle$ or $P|x\rangle$?

presumably $P|p\rangle = p|p\rangle$

momentum eigenstate

We start with $T(dx)|\psi\rangle$:

$$\underbrace{T(dx)|\psi\rangle}_{} = \left(\mathbb{1} - i\frac{dx}{\hbar}P\right)|\psi\rangle$$

insert $\mathbb{1}$

$$\text{LHS} = \int dx'\, \underbrace{T(dx)|x'\rangle}_{}\langle x'|\psi\rangle = \int dx'\, |x'+dx\rangle\langle x'|\psi\rangle$$

change of variable $x'' = x' + dx$, $dx'' = dx'$

$$= \int dx''\, |x''\rangle \underbrace{\langle x''-dx|\psi\rangle}_{\psi(x''-dx)}$$

Taylor expand

$$= \int dx''\, |x''\rangle \left(\psi(x'') - \frac{\partial}{\partial x''}\psi(x'')\,dx + \frac{1}{2!}\frac{\partial^2}{\partial x''^2}\psi(x'')\,dx^2 + \cdots\right)$$

$$\simeq \underbrace{\int dx''\, |x''\rangle\langle x''|}_{\mathbb{1}}\psi\rangle - dx\int dx''\, |x''\rangle \frac{\partial}{\partial x''}\langle x''|\psi\rangle + \cdots$$

$$\simeq |\psi\rangle - dx\left(\int dx''\, |x''\rangle \frac{\partial}{\partial x''}\langle x''|\right)|\psi\rangle + \cdots$$

$$\text{RHS} = \left(\mathbb{1} - i\frac{dx}{\hbar}P\right)|\psi\rangle = |\psi\rangle - dx\,\frac{i}{\hbar}P|\psi\rangle$$

$$\text{LHS} = \text{RHS} \Rightarrow -\int dx''\, |x''\rangle \frac{\partial}{\partial x''}\langle x''| = -\frac{i}{\hbar}P$$

$$\Leftrightarrow \boxed{P = -i\hbar\int dx\, |x\rangle\frac{\partial}{\partial x}\langle x|}$$

momentum operator in the $\{|x\rangle\}$ basis

Example: $$\langle\psi|P|\psi\rangle = \int dx\, \underbrace{\langle\psi|x\rangle}_{\langle x|\psi\rangle^* = \psi^*(x)} \underbrace{\left(-i\hbar\frac{\partial}{\partial x}\right)\underbrace{\langle x|\psi\rangle}_{\psi(x)}}_{\frac{\partial}{\partial x}\psi(x)}$$