Thursday, September 17, 2026

Q: How do we calculate $P|\psi\rangle$ or $\underbrace{P|x\rangle}$?

$\hookrightarrow$ and we require $P|p\rangle = p|p\rangle$

momentum eigenstate

A: We start with $T(dx)|\psi\rangle$:

$$\underbrace{T(dx)|\psi\rangle}_{\downarrow \text{ insert } \mathbb{1}} = \left(\mathbb{1} - i\frac{dx}{\hbar}P\right)|\psi\rangle$$

$$LHS = \int dx'\, \underbrace{T(dx)|x'\rangle}\langle x'|\psi\rangle = \int dx'\, |x'+dx\rangle\langle x'|\psi\rangle$$

change of variable $x'' = x' + dx$, $dx'' = dx'$

$$= \int dx''\, |x''\rangle \underbrace{\langle x'' - dx|\psi\rangle}_{\psi(x''-dx)}$$

$\hookrightarrow$ Taylor expand

$$= \int dx''\, |x''\rangle \left(\psi(x'') - \frac{\partial}{\partial x''}\psi(x'')\,dx + \frac{1}{2!}\frac{\partial^2}{\partial x''^2}\psi(x'')\,dx^2 + \cdots\right)$$

$$\simeq \underbrace{\int dx''\, |x''\rangle \overbrace{\langle x''|}^{\psi(x'')}}_{\mathbb{1}}\psi\rangle - dx\int dx''\, |x''\rangle \frac{\partial}{\partial x''}\langle x''|\psi\rangle + \cdots$$

$$\simeq |\psi\rangle - dx\left(\int dx''\, |x''\rangle \frac{\partial}{\partial x''}\langle x''|\right)|\psi\rangle + \cdots$$

$$RHS = \left(\mathbb{1} - i\frac{dx}{\hbar}P\right)|\psi\rangle = |\psi\rangle - dx\,\frac{i}{\hbar}P|\psi\rangle$$

$$LHS = RHS \Rightarrow -\int dx''\, |x''\rangle \frac{\partial}{\partial x''}\langle x''| = -\frac{i}{\hbar}P$$

$$\Leftrightarrow \boxed{P = -i\hbar \int dx\, |x\rangle \frac{\partial}{\partial x}\langle x|}$$

momentum operator in the $\{|x\rangle\}$ basis

Example: $$\langle\varphi|P|\psi\rangle = \int dx\, \underbrace{\langle\varphi|x\rangle}_{\langle x|\varphi\rangle^* = \varphi^*(x)} \underbrace{\left(-i\hbar\frac{\partial}{\partial x}\right)\underbrace{\langle x|\psi\rangle}_{\psi(x)}}_{\frac{\partial}{\partial x}\psi(x)}$$

$$\Rightarrow \langle \varphi | P | \psi \rangle = \int dx\, \varphi^*(x) \left[-i\hbar \frac{\partial}{\partial x} \psi(x) \right]$$

$$= -i\hbar \int dx \left[\varphi^*(x) \frac{\partial}{\partial x} \psi(x) \right]$$

Q: What are the momentum eigenstates $|p\rangle$ (i.e. $P|p\rangle = p|p\rangle$) in the position eigenbasis $\{|x\rangle\}$?

A:

$$P|p\rangle = p|p\rangle$$

$\psi_p(x')$ = momentum eigenstate in $\{|x'\rangle\}$ basis

$$\Rightarrow \left[-i\hbar \int dx' |x'\rangle \frac{\partial}{\partial x'} \langle x'| \right] |p\rangle = p|p\rangle$$

$$\Leftrightarrow -i\hbar \int dx' |x'\rangle \frac{\partial}{\partial x'} \psi_p(x') = p|p\rangle$$

multiply on left by $\langle x|$

$$\Rightarrow -i\hbar \int dx \underbrace{\langle x | x' \rangle}_{\delta(x-x')} \frac{\partial}{\partial x'} \psi_p(x') = p \underbrace{\langle x | p \rangle}_{\psi_p(x)}$$

$$\Leftrightarrow -i\hbar \frac{\partial}{\partial x} \psi_p(x) = p\, \psi_p(x)$$

$$\Leftrightarrow \frac{\partial}{\partial x} \psi_p(x) = \frac{ip}{\hbar} \cdot \psi_p(x)$$

$$\Rightarrow \boxed{\psi_p(x) = A\, e^{i\frac{p}{\hbar}x}} = \langle x | p \rangle$$

$\Rightarrow |p\rangle$ is a plane wave

Amplitude must be specified for an orthonormal basis

$\Rightarrow \{|p\rangle\}$ eigenbasis is the set of all planewaves orthonormality of the $\{|p\rangle\}$ eigenbasis (as with the $\{|x\rangle\}$ eigenbasis) requires that

$$\langle p | p' \rangle = \delta(p - p')$$

To determine the normalization amplitude "A", we note the following:

$$\underbrace{\langle x | x' \rangle}_{\delta(x-x')} = \langle x | \underbrace{\left(\int dp' \, |p'\rangle\langle p'| \right)}_{= \mathbb{1} \text{ (using } \{|p\rangle\} \text{ eigenbasis completeness)}} |x'\rangle$$

$$\Leftrightarrow \quad \delta(x - x') = \int dp' \Big[\underbrace{\langle x | p' \rangle}_{A e^{ip'x/\hbar}} \underbrace{\langle p' | x' \rangle}_{A^* e^{-ip'x'/\hbar}} \Big]$$

$$= |A|^2 \int dp' \exp\left[\frac{ip'(x - x')}{\hbar} \right]$$

$$\boxed{\begin{array}{l} \text{Special property of Dirac delta \& Fourier transforms} \\ \displaystyle\int_{-\infty}^{+\infty} e^{ik(x-x')} \, dk = 2\pi \, \delta(x - x') \end{array}}$$

see PHY 603: grad Math Physics

change of variable $k' = p'/\hbar$, $dk' = dp'/\hbar$

$$\text{thus } \delta(x - x') = |A|^2 \hbar \underbrace{\int_{-\infty}^{+\infty} dk' \exp[ik(x - x')]}_{2\pi\,\delta(x-x')}$$

$$\Leftrightarrow \delta(x - x') = |A|^2 2\pi\hbar \, \delta(x - x') \quad \Rightarrow \quad |A|^2 = \frac{1}{2\pi\hbar}$$

We choose $A = \frac{1}{\sqrt{2\pi\hbar}}$

thus $\boxed{\langle x|p\rangle = \psi_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{i\frac{px}{\hbar}}}$

<u>x-representation of a state $|\psi\rangle$</u>

also called q-representation or configuration space representation

$$\langle x|\psi\rangle = \psi(x)$$

<u>p-representation of a state $|\psi\rangle$</u> (or momentum space representation)

$$\langle p|\psi\rangle = \psi(p)$$

Q: How does one convert between $\psi(x)$ and $\psi(p)$?

A: Use the identity operator(s)?

$$\psi(x) = \langle x|\psi\rangle = \langle x|\left(\int dp\, |p\rangle\langle p|\right)|\psi\rangle$$

$$= \int dp\, \underbrace{\langle x|p\rangle}_{\frac{1}{\sqrt{2\pi\hbar}} e^{i\frac{px}{\hbar}}} \underbrace{\langle p|\psi\rangle}_{\psi(p)}$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int dp \left[\psi(p)\, e^{i\frac{px}{\hbar}}\right]$$

$$\Rightarrow \boxed{\begin{aligned} \psi(x) &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} dp \left[\psi(p)\, e^{i\frac{px}{\hbar}} \right] \\ \langle x|\psi\rangle &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} dp \left[\langle p|\psi\rangle\, e^{i\frac{px}{\hbar}} \right] \end{aligned}}$$

you convert ~~from~~ from $\psi(p)$ to $\psi(x)$ with a "Fourier transform".

Likewise,

$$\psi(p) = \langle p|\psi\rangle = \langle p| \left(\int dx\, |x\rangle\langle x| \right) |\psi\rangle$$

$$= \int dx\, \underbrace{\langle p|x\rangle}_{\frac{1}{\sqrt{2\pi\hbar}} \exp\left(-\frac{ipx}{\hbar}\right)}\, \underbrace{\langle x|\psi\rangle}_{\psi(x)}$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int dx \left[\psi(x)\, e^{-i\frac{px}{\hbar}} \right]$$

$$\Rightarrow \boxed{\begin{aligned} \psi(p) &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} dx \left[\psi(x)\, e^{-i\frac{px}{\hbar}} \right] \\ \langle p|\psi\rangle &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} dp \left[\langle x|\psi\rangle\, e^{-i\frac{px}{\hbar}} \right] \end{aligned}}$$

you convert from $\psi(x)$ to $\psi(p)$ with an "inverse Fourier transform"

Time Evolution Operator

Similar to the spatial translation operator $T(\Delta x)|x\rangle = |x+\Delta x\rangle$ (unitary)