Tuesday, September 22, 2026

Similar to the spatial translation operator $T(\Delta x)|x\rangle = |x+\Delta x\rangle$ (unitary),

We can construct a unitary operator for time evolution:

$$\mathcal{U}(t,t_0)\,|\psi;t_0\rangle = |\psi;t\rangle$$

note: $|\psi;t_0\rangle = \sum_i c_i(t_0)\underbrace{|i\rangle}$

↳ some eigenbasis $\{|i\rangle\}$ e.g., $\{|x\rangle\}$ or $\{|p\rangle\}$.

and $|\psi;t\rangle = \sum_i \underbrace{c_i(t)}\,\underbrace{|i\rangle}$

$c_i(t) \neq c_i(t_0)$

⚠ we did not time evolve the eigenkets $\{|i\rangle\}$.
[we could have, but we choose not to]

the operator for an infinitesimal time step dt of time evolution is

$$\boxed{\mathcal{U}(t_0+dt,t_0) = \mathbb{1} - i\frac{dt}{\hbar}H}$$

H = Hamiltonian for the system

We can now look at the time evolution of $\mathcal{U}(t,t_0)$:

$$\mathcal{U}(t+dt,t_0) = \mathcal{U}(t+dt,t)\,\underbrace{\mathcal{U}(t,t_0)}_{t-t_0 \text{ is not infinitesimal}} = \left(\mathbb{1} - \frac{i\,dt}{\hbar}H\right)\mathcal{U}(t,t_0)$$

$$= \mathcal{U}(t,t_0) - \frac{i\,dt}{\hbar}H\,\mathcal{U}(t,t_0)$$

$$\Longleftrightarrow \frac{\mathcal{U}(t+dt,t_0) - \mathcal{U}(t,t_0)}{dt} = \frac{-i}{\hbar}H\,\mathcal{U}(t,t_0)$$

$$\Longleftrightarrow \boxed{i\hbar \frac{\partial}{\partial t} U(t,t_0) = H\, U(t,t_0)}$$

Schrödinger equation for the time evolution operator $U(t,t_0)$.

multiply on the right by a ket $|\psi;t_0\rangle$

$$i\hbar \frac{\partial}{\partial t} \underbrace{U(t,t_0)|\psi;t_0\rangle}_{|\psi;t_0\to t\rangle} = H \underbrace{U(t,t_0)|\psi;t_0\rangle}_{|\psi;t_0\to t\rangle}$$

$$\Rightarrow \boxed{i\hbar \frac{\partial}{\partial t} \underbrace{|\psi;t_0\to t\rangle}_{|\psi(t)\rangle} = H \underbrace{|\psi;t_0\to t\rangle}_{|\psi(t)\rangle}}$$

Schrodinger equation for the ket $|\psi;t_0\rangle$

In the case of a <u>time-independent</u> Hamiltonian H, then

$$U(t,t_0) = \exp\left[-\frac{i}{\hbar} H (t-t_0)\right] = \exp\left[-\frac{i(t-t_0)}{\hbar} H\right]$$

H is an operator

$$= \lim_{N\to\infty} \left[1 - \frac{i}{\hbar}\frac{H(t-t_0)}{N}\right]^N$$

recall: $e^A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$

note: $A^0 = \mathbb{1}$

$$= \mathbb{1} + \left(\frac{-i(t-t_0)}{\hbar}\right) H + \frac{1}{2!}\left(\frac{-i(t-t_0)}{\hbar}\right)^2 H^2 + \frac{1}{3!}\left(\frac{-i(t-t_0)}{\hbar}\right)^3 H^3 + \ldots$$

⚠ Calculating $\exp\left[-i\frac{(t-t_0)}{\hbar} H\right]$ is very difficult, unless you go the eigenbasis of H... then it's relatively easy.

The eigenstates of the Hamiltonian operator simplify these equations / expressions significantly:

$$\underset{\text{Hermitian}}{H} |\psi_E\rangle = \underset{\substack{\text{Eigenvalue} \\ = \text{Eigenenergy } (E \in \mathbb{R})}}{E} \underbrace{|\psi_E\rangle}_{\text{eigenstate} \,=\, \text{stationary state}}$$

potentially hard to calculate

$$\text{time evolution:} \quad \mathcal{U}(t,t_0)|\psi_E\rangle = e^{-i\frac{(t-t_0)}{\hbar}H}|\psi_E\rangle$$

easy to calculate

$$= e^{-i\left(\frac{t-t_0}{\hbar}\right)E}|\psi_E\rangle$$

$$= |\psi_E(t)\rangle$$

Energy eigenstates [i.e., stationary states] do <u>not</u> change in time, except for an <u>overall phase factor</u>

Likewise, the Schrodinger equation is simple to solve with the energy eigenstates:

$$i\hbar\frac{\partial}{\partial t}|\psi_E(t)\rangle = H|\psi_E(t)\rangle = E|\psi_E(t)\rangle$$

$$\Rightarrow \quad |\psi_E(t)\rangle = e^{-i\frac{(t-t_0)}{\hbar}E}|\psi(t_0)\rangle$$

<u>Note</u>: Figuring out (i.e., determining) the energy eigenstates is not-so-easy.

Once the energy eigenbasis is determined $\{|\psi_{E_i}\rangle\}$, then the time evolution for any state can be easily determined:

$$\boxed{\begin{aligned} &\text{If} \quad |\psi(t=0)\rangle = \sum_n c_n |\psi_{E_n}\rangle \\ &\Rightarrow \text{then} \quad |\psi(t)\rangle = \sum_n \underbrace{c_n e^{-i\frac{E_n}{\hbar}t}}_{c_n(t)} |\psi_{E_n}\rangle \end{aligned}}$$

"Good Quantum Number"

Consider an operator A (with eigenstates $\{|a_i\rangle\}$ and eigenvalues $\{a_i\}$) that commutes with H: $[H, A] = 0$ and $A|a_i\rangle = a_i|a_i\rangle$

then the time evolution of the $|a_i\rangle$ eigenstates is simple

If $|a_i(t)\rangle = U(t)\underbrace{|a_i\rangle}_{|a_i(t=0)\rangle}$, then we can calculate $A|a_i(t)\rangle$

$$A|a_i(t)\rangle = \underbrace{A\, e}_{[H,A]=0}{}^{-iHt/\hbar}|a_i\rangle = e^{-iHt/\hbar}\underbrace{A|a_i\rangle}_{a_i|a_i\rangle}$$

$$= a_i\, e^{-iHt/\hbar}|a_i\rangle$$

$$= a_i|a_i(t)\rangle$$

$\Rightarrow$ $\boxed{|a_i(t)\rangle \text{ remains an eigenstate of } A}$ $\Rightarrow$ $\boxed{a_i \text{ is a "good quantum number"}}$

Note: $|a_i\rangle$ is also an eigenstate of H.

Schrodinger Wave Equation

The Hamiltonian for a single particle (of mass m) moving in an external potential $V(x)$ [$V(x)$ = potential energy] is

$$H = \frac{P^2}{2m} + V(X) = \frac{1}{2m}P^2 + V(X)$$

P operator $\qquad$ X operator

The Schrodinger equation becomes

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H |\psi(t)\rangle$$

In position space (X-representation/basis), we get

$$\langle x| \, i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \langle x| H |\psi(t)\rangle$$

$$\Leftrightarrow \quad i\hbar \frac{\partial}{\partial t} \underbrace{\langle x|\psi(t)\rangle}_{\psi(x,t)} = \underbrace{\langle x| \frac{1}{2m} P^2 |\psi(t)\rangle}_{\frac{1}{2m}\langle x| P \cdot P |\psi(t)\rangle} + \underbrace{\langle x| V(X) |\psi(t)\rangle}_{V(x)\langle x|\psi(t)\rangle}$$

operator X

$= \langle x| V(x)$

scalar x

$$P = -i\hbar \int dx' |x'\rangle \frac{\partial}{\partial x'} \langle x'|$$

$$P = -i\hbar \int dx'' |x''\rangle \frac{\partial}{\partial x''} \langle x''|$$

$$\frac{1}{2m}\langle x| P \cdot P |\psi(t)\rangle = \frac{-\hbar^2}{2m} \left(\int dx'' \underbrace{\langle x|x''\rangle}_{\delta(x-x'')} \frac{\partial}{\partial x''} \underbrace{\langle x''| \int dx' |x'\rangle}_{\langle x''|x'\rangle = \delta(x''-x')} \underbrace{\frac{\partial}{\partial x'} \langle x'| \right) |\psi(t)\rangle}_{\frac{\partial}{\partial x'} \psi(x',t)}$$

$$= \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi(x,t)$$

$$\Leftrightarrow \quad \boxed{i\hbar \frac{\partial}{\partial t} \psi(x,t) = \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi(x,t) + V(x)\,\psi(x,t) = \overbrace{\left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right]}^{\text{“}H\text{”}} \psi(x,t)}$$

Schrödinger Wave Equation

This equation is much easier to solve, if we look for the stationary state ~~st~~ solutions: $\psi_E(x;t) = e^{-i\frac{Et}{\hbar}} \psi_E(x)$

Where $\begin{Bmatrix} \psi_E(x) \\ E \end{Bmatrix}$ satisfies: $\underbrace{\boxed{\Big[\underbrace{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)}_{\text{"}H\text{"}}\Big]\psi_E(x) = E\psi_E(x)}}_{\text{Eigenenergy-eigenstate equation}}$

Note 1: We require that $\int_{-\infty}^{+\infty} \psi_E^*(x)\,\psi_E(x)\,dx = 1$ ← to be revisited for a continuous eigenbasis

Note 2: $\{|\psi_{E_n}\rangle\}$ form an eigenbasis: $\sum_n |\psi_{E_n}\rangle\langle\psi_{E_n}| = \mathbb{1}$

or $\int dn\, |\psi_{E_n}\rangle\langle\psi_{E_n}| = \mathbb{1}$

orthonormality: $\langle\psi_{E_n}|\psi_{E_{n'}}\rangle = \begin{cases} \delta_{nn'} & \text{discrete} \\ \delta(n-n') & \text{continuous} \end{cases}$

Heisenberg "Picture" of time evolution

Schrodinger picture: $\begin{cases} \text{Operator/observable are time-independent} \\ \text{kets evolve in time} \end{cases}$

Heisenberg picture: $\begin{cases} \text{Operators evolve in time} \\ \text{kets are time independent} \end{cases}$

Consider $\langle\psi_1(t)|\,A\,|\psi_2(t)\rangle = \langle\psi_1|\,e^{i\frac{Ht}{\hbar}}\,A\,e^{-i\frac{Ht}{\hbar}}\,|\psi_2\rangle$

In the Heisenberg picture $A \to \boxed{A(t) = e^{i\frac{Ht}{\hbar}}\,A\,e^{-i\frac{Ht}{\hbar}}}$