Thursday, September 24, 2026

The Schrodinger equation, $i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle$, can be written in the $x$ representation as:

$$\boxed{\begin{aligned} i\hbar \frac{\partial}{\partial t}\psi(x,t) &= \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) + V(x)\psi(x,t) \\ &= \underbrace{\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]}_{\text{"}H\text{"}}\psi(x,t) \end{aligned}}$$

Schrodinger wave equation

This equation is much easier to solve, if we look for the stationary state solutions: $\psi_E(x,t) = e^{-i\frac{E}{\hbar}t}\psi_E(x)$

where $\left\{\begin{matrix}\psi_E(x) \\ E\end{matrix}\right\}$ satisfies $\boxed{\underbrace{\left[\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]}_{\text{"}H\text{"}}\psi_E(x) = E\psi_E(x)}$

Eigenenergy - eigenstate equation

time-independent Schrodinger eq.

Note 1: We require that $\int_{-\infty}^{+\infty} \psi_E^*(x)\psi(x)\,dx = 1$ ← to be revisited for a continuous eigenbasis

Note 2: $\{|\psi_{E_n}\rangle\}$ form an eigenbasis: $\left\{\begin{matrix}\sum_n |\psi_{E_n}\rangle\langle\psi_{E_n}| = \mathbb{1} \\ \int dn\, |\psi_{E_n}\rangle\langle\psi_{E_n}| = \mathbb{1}\end{matrix}\right.$

Orthonormality: $\langle\psi_{E_n}|\psi_{E_{n'}}\rangle = \begin{cases}\delta_{nn'} & \text{discrete} \\ \delta(n-n') & \text{continuous basis}\end{cases}$

# Heisenberg "picture" of time evolution

Schrodinger picture : $\begin{cases} \text{Operator/observable are time-independent} \\ \text{kets evolve in time} \end{cases}$

Heisenberg picture : $\begin{cases} \text{Operators evolve in time} \\ \text{kets are time independent} \end{cases}$

Consider $\langle \psi_1(t) | A | \psi_2(t) \rangle = \langle \psi_1 | e^{\frac{iHt}{\hbar}} A \, e^{-\frac{iHt}{\hbar}} | \psi_2 \rangle$

H = Heisenberg

In the Heisenberg picture $A \rightarrow \boxed{A^H(t) = e^{\frac{iHt}{\hbar}} A^S e^{-\frac{iHt}{\hbar}}}$

S = Schrodinger

with time evolution operators: $\begin{cases} U = e^{-\frac{iHt}{\hbar}} \\ U^+ = e^{+\frac{iHt}{\hbar}} \end{cases}$ | note: $U^+ H U = H$ $\Rightarrow H^H = H^S$

$$\frac{\partial}{\partial t} A^H = \frac{\partial}{\partial t}\left(U^+ A^S U\right) = \underbrace{\left(\frac{\partial}{\partial t} U^+\right)}_{-\frac{1}{i\hbar} U^+ H} A^S U + U^+ A \underbrace{\left(\frac{\partial}{\partial t} U\right)}_{\frac{1}{i\hbar} H U}$$

"Schrodinger eq. for the time evolution operator" see Sept. 22 lecture

$$= -\frac{1}{i\hbar} U^+ H A^S U + \frac{1}{i\hbar} U^+ A H U$$

$\mathbb{1} = U^+ U = U U^+$

$$= -\frac{1}{i\hbar} \underbrace{U^+ H U}_{H} \underbrace{U^+ A^S U}_{A^H} + \frac{1}{i\hbar} \underbrace{U^+ A U}_{A^H} \underbrace{U^+ H U}_{H}$$

$$= \frac{1}{i\hbar}\left(A^{(H)} H - H A^{(H)}\right) \Rightarrow \boxed{i\hbar \frac{\partial}{\partial t} A^{(H)}(t) = \left[A^{(H)}(t), H\right]}$$

Heisenberg time evolution equation

Note: In classical physics, the equation of motion can be written in the form
for a quantity A

$$\frac{d}{dt} A_{classical} = \underbrace{[\![A_{classical}, H]\!]}_{\text{Poisson braket}}$$

$$\underbrace{\frac{1}{i\hbar}[\ ,\ ]}_{\text{quantum}} \longrightarrow \underbrace{[\![\ ,\ ]\!]}_{\text{classical}}$$

## Ehrenfest's Theorem

We consider the Hamiltonian for a particle in a potential $V(x)$:

$$H = \frac{1}{2m} P^2 + V(X)$$

drop the "(H)" identifier

$$\frac{\partial}{\partial t} X^{(H)} = \frac{1}{i\hbar}[X^{(H)}, H] = \frac{1}{i\hbar}\underbrace{\left[X, \frac{1}{2m}P^2 + V(X)\right]}_{\frac{1}{2m}\underbrace{[X, P^2]}_{i\hbar 2P} + \underbrace{[X, V(X)]}_{=0}}$$

see P.S. # Q4(a,b)

$$\Rightarrow \frac{\partial}{\partial t} X = \frac{1}{\cancel{i\hbar}} \frac{1}{\cancel{2}m} \cancel{i\hbar}\, \cancel{2} P \iff \frac{\partial}{\partial t} X^{(H)} = \frac{1}{m} P^{(H)}$$

In terms of an expectation value for a state $|\psi\rangle$, we get

what get's measured on average

$$\boxed{\frac{\partial}{\partial t}\langle X\rangle_\psi = \frac{1}{m}\langle P\rangle_\psi} \quad (i)$$

i.e. $\frac{dx}{dt} = v$

drop "(H)" identifier

$$\frac{\partial}{\partial t} P^{(H)} = \frac{1}{i\hbar}\left[P^{(H)}, H\right] = \frac{1}{i\hbar}\left[P, \frac{1}{2m}P^2 + V(X)\right]$$

$$= \frac{1}{\cancel{i\hbar}}\left\{ \underbrace{\left[P, \frac{1}{2m}P^2\right]}_{=0} + \underbrace{\left[P, V(X)\right]}_{-\cancel{i\hbar}\frac{\partial}{\partial X}V(X)} \right\}$$

$$\Rightarrow \quad \frac{\partial}{\partial t} P^{(H)} = -\frac{\partial}{\partial X} V(X)^{(H)}$$

In terms of an expectation value for a state $|\psi\rangle$, we have

$$\boxed{\frac{\partial}{\partial t}\langle P\rangle_\psi = -\left\langle \frac{\partial}{\partial X} V(X)\right\rangle_\psi} \quad (ii)$$

$$(i) \,\&\, (ii) \Rightarrow \quad \boxed{m\frac{\partial^2}{\partial t^2}\langle X\rangle_\psi = -\left\langle \frac{\partial}{\partial X} V(X)\right\rangle_\psi}$$

i.e. Classical equation of motion

$\Rightarrow$ On average QM gives classical physics

(though this result depends on using a "classical" $|\psi\rangle$)

## Conservation of Probability

We consider the Schrodinger wave equation:

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) + V(x)\,\psi(x,t)$$

$p(x,t) = |\psi(x,t)|^2$ can be interpreted as the probility density to find the particle at position $x$ at time $t$

$$\Rightarrow \quad dp(x,t) = |\psi(x,t)|^2 \, dx\, dt$$

We rename to $\rho(x,t) = |\psi(x,t)|^2 = \psi^*(x,t)\,\psi(x,t)$

Let's look at the time evolution of $\rho(x,t)$:

$$\frac{\partial}{\partial t}\rho(x,t) = \left(\frac{\partial}{\partial t}\psi^*(x,t)\right)\psi(x,t) + \psi^*(x,t)\left(\frac{\partial}{\partial t}\psi(x,t)\right)$$

$$= \frac{1}{i\hbar}\Bigg[-\underbrace{\left(-i\hbar\frac{\partial}{\partial t}\psi^*\right)}_{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi^* + \cancel{V(x)\psi^*}}\psi + \psi^*\underbrace{\left(i\hbar\frac{\partial}{\partial t}\psi\right)}_{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi + \cancel{V(x)\psi}}\Bigg]$$

$$= \frac{1}{i\hbar}\left(\frac{-\hbar^2}{2m}\right)\Bigg[\underbrace{\psi^*\frac{\partial^2}{\partial x^2}\psi - \psi\frac{\partial^2}{\partial x^2}\psi^*}\Bigg]$$

$$= \frac{\partial}{\partial x}\left[\psi^*\frac{\partial}{\partial x}\psi - \psi\frac{\partial}{\partial x}\psi^*\right]$$

$$= \cancel{\frac{\partial}{\partial x}\psi^*\frac{\partial}{\partial x}\psi} + \psi^*\frac{\partial^2}{\partial x^2}\psi - \cancel{\frac{\partial}{\partial x}\psi\frac{\partial}{\partial x}\psi^*} - \psi\frac{\partial^2}{\partial x^2}\psi^*$$

$$= -\frac{\partial}{\partial x}\Bigg[\underbrace{\frac{-i\hbar}{2m}\left(\psi^*\frac{\partial}{\partial x}\psi - \psi\frac{\partial}{\partial x}\psi^*\right)}_{= J(x,t)}\Bigg]$$

$= J(x,t)$ (definition)

$=$ Probability current density $=$ probability flux

$$= \left(\frac{\hbar}{m}\right)\mathrm{Im}\left(\psi^*\frac{\partial}{\partial x}\psi\right)$$

$$\Rightarrow \quad \boxed{\frac{\partial \rho(x,t)}{\partial t} = -\frac{\partial}{\partial x}J(x,t) \iff \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}J = 0}$$

Continuity equation = conservation of probability