

Tuesday, September 29, 2026

Dynamics of 2-level systems

We consider a generic 2-level quantum mechanical system with 2 distinct eigenstates and eigenenergies

The diagram shows two energy levels: a lower level labeled $|g\rangle, E_g$ and an upper level labeled $|e\rangle, E_e$. A red double-headed arrow between them is labeled $E_{eg} = E_e - E_g$. To the right, the Hamiltonian matrix is given as:

$$\langle g | H = \begin{bmatrix} E_g & 0 \\ 0 & E_e \end{bmatrix} \begin{matrix} |g\rangle \\ |e\rangle \end{matrix}$$

(I) Basic time-dependence

What happens when you let an arbitrary state evolve in time?

Easy: $|\psi\rangle = \alpha|g\rangle + \beta|e\rangle \Rightarrow |\psi(t)\rangle = \alpha e^{-i\frac{E_g t}{\hbar}} |g\rangle + \beta e^{-i\frac{E_e t}{\hbar}} |e\rangle$

note: $|\alpha|^2 + |\beta|^2 = 1$
(normalization)

$$= e^{-i\frac{E_g t}{\hbar}} \left[\alpha |g\rangle + \beta e^{-i\frac{(E_e - E_g)t}{\hbar}} |e\rangle \right]$$

overall phase | relative phase
 $\hookrightarrow$ not important | $\omega_{eg} = \frac{E_{eg}}{\hbar}$

$$\left. \begin{aligned} P(|g\rangle)_{|\psi(t)\rangle} &= |\langle g | \psi(t) \rangle|^2 = |\alpha|^2 \\ P(|e\rangle)_{|\psi(t)\rangle} &= |\langle e | \psi(t) \rangle|^2 = |\beta|^2 \end{aligned} \right\} \begin{array}{l} \text{no measurable} \\ \text{time-dependence} \end{array}$$

more interesting:

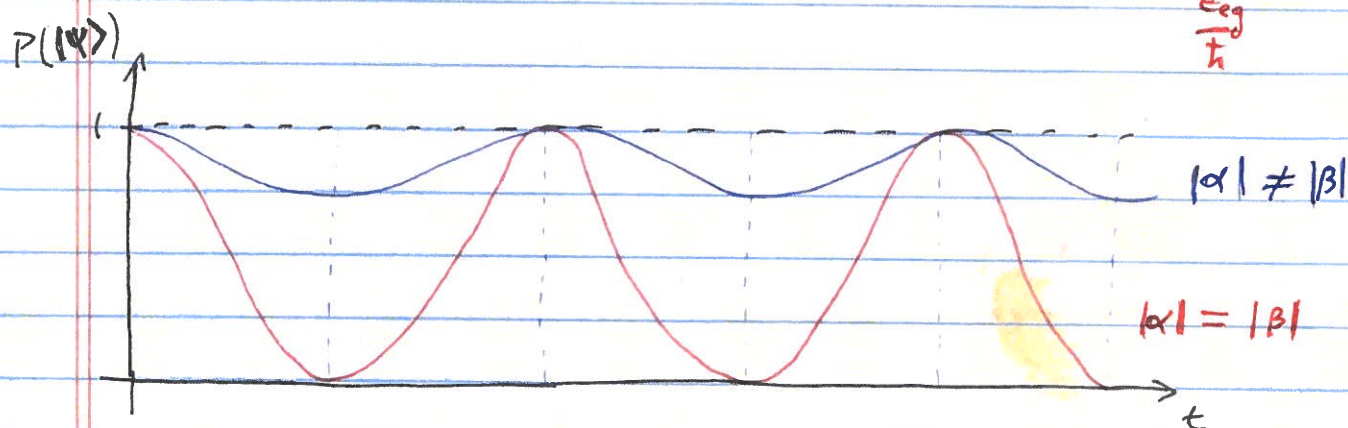
what is the probability to remain in the original state

$$P(|\psi\rangle)_{|\psi(t)\rangle} = |\langle\psi|\psi(t)\rangle|^2$$

$$= \left| \left(\alpha^* \langle g| + \beta^* \langle e| \right) \left(\alpha e^{-i \frac{E_g}{\hbar} t} |g\rangle + \beta e^{-i \frac{E_e}{\hbar} t} |e\rangle \right) \right|^2$$

(some algebra)

$$= \frac{1}{2} \left[1 + (|\alpha|^2 - |\beta|^2)^2 \right] + \frac{1}{2} \left[1 - (|\alpha|^2 - |\beta|^2)^2 \right] \cos \left(\underbrace{\omega_{eg}}_{\frac{E_e - E_g}{\hbar}} t \right)$$



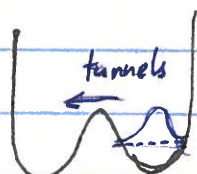
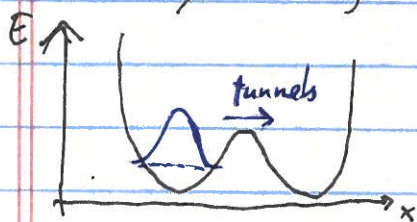
There is a time-dependence if you look another basis

examples:

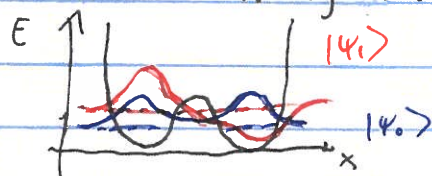
1. AC Josephson effect (or Josephson oscillations)

consider a double well potential

if you put your particles (e^- 's, atoms, etc...) in one well, then you will get oscillation between the two wells



why? measurements are not done in eigenbasis



Superconductor: SC-I-SC junction or BEC in a double well

If we consider just the ground state $|\psi_0\rangle$ and 1st excited state $|\psi_1\rangle$, then

$$\left. \begin{aligned} |\text{left}\rangle &= |\psi_0\rangle + |\psi_1\rangle \\ |\text{right}\rangle &= |\psi_0\rangle - |\psi_1\rangle \end{aligned} \right\} \text{orthogonal}$$

↑ measurement basis
↑ energy eigenbasis

note: $|\psi_0\rangle$ & $|\psi_1\rangle$ can be single particle states or many-body eigenstates.

Rabi Régime (non-interacting particles)
 ↑ Josephson Régime (weak inter-particle interactions)

2. Neutrino oscillations

Neutrino's come in 3 flavors ν_e electron- $\nu \rightarrow |\psi_{\nu_e}\rangle$
 (+ anti-particles) ν_μ muon- $\nu \rightarrow |\psi_{\nu_\mu}\rangle$
 ν_τ tau- $\nu \rightarrow |\psi_{\nu_\tau}\rangle$

These are the measurement basis, but not the energy eigenbasis (i.e. mass basis):

Relativistic Energy: $E = \sqrt{p^2 c^2 + m_0^2 c^4} \approx pc + \frac{1}{2} m_0^2 \frac{c^3}{p}$
 (for neutrinos $E_{\text{kinetic}} \gg E_{\text{rest}} = m_0 c^2$)

$$\begin{aligned} |\psi_{\nu_{m_1}}\rangle, E_1 &\approx pc + \frac{1}{2} m_1^2 \frac{c^3}{p} \\ |\psi_{\nu_{m_2}}\rangle, E_2 &\approx pc + \frac{1}{2} m_2^2 \frac{c^3}{p} \\ |\psi_{\nu_{m_3}}\rangle, E_3 &\approx pc + \frac{1}{2} m_3^2 \frac{c^3}{p} \end{aligned}$$

The time evolution of the neutrino wavefunction is determined in the mass/energy eigenbasis

$$|\psi(t)\rangle = \alpha e^{-i \frac{E_1 t}{\hbar}} |\psi_{\nu_{m_1}}\rangle + \beta e^{-i \frac{E_2 t}{\hbar}} |\psi_{\nu_{m_2}}\rangle + \gamma e^{-i \frac{E_3 t}{\hbar}} |\psi_{\nu_{m_3}}\rangle$$

with for example $|\psi_{\nu_e}\rangle = \alpha_e |\psi_{\nu_{m_1}}\rangle + \beta_e |\psi_{\nu_{m_2}}\rangle + \gamma_e |\psi_{\nu_{m_3}}\rangle$
 change of basis matrix
 = Pontecorvo-Maki-Nakagawa-Sakata matrix (P)MNS matrix

We see oscillation because measurement and mass basis are different
 ↳ one can only see oscillations proportional to the difference of masses squared: $\Delta E \propto \Delta m_{ij}^2 = m_i^2 - m_j^2$

3. Rabi oscillations in the dressed atom picture

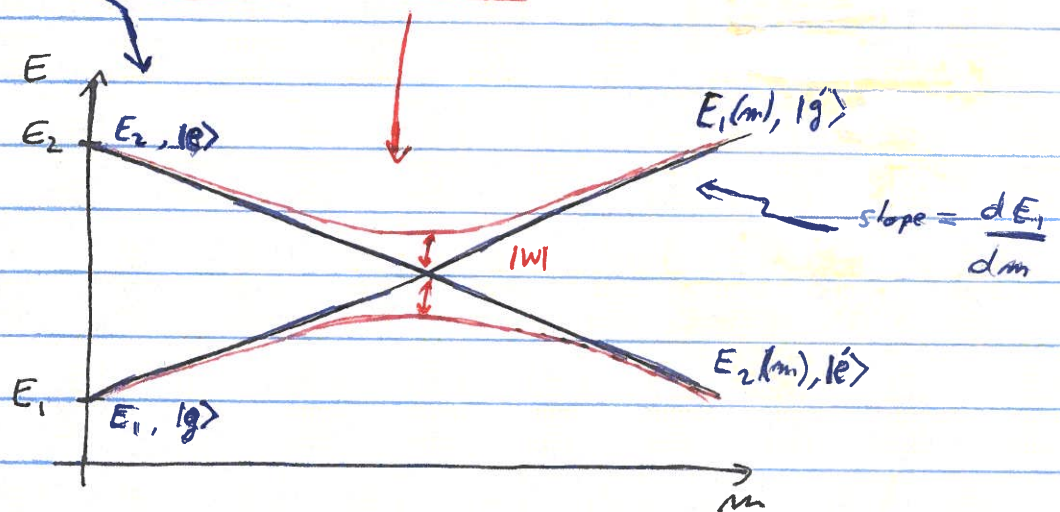
atom + simplified quantized E-M field.

(II) Landau - Zener Transitions

We consider a 2-level system with an off-diagonal interaction W (inter-state)

in which the base energies are varied through each other:

$$H = \begin{matrix} & |g\rangle & |e\rangle \\ \begin{matrix} \langle g| \\ \langle e| \end{matrix} & \begin{bmatrix} E_1(m) & 0 \\ 0 & E_2(m) \end{bmatrix} & + \begin{bmatrix} 0 & W \\ W^* & 0 \end{bmatrix} \end{matrix}$$



Effect of W

There is an avoided level crossing, and $E_2(m \rightarrow 0)$ is connected to $E_1(m \rightarrow \text{large}), |g'\rangle$, and $E_1(m \rightarrow 0), |g\rangle$ is connected to $E_2(m \rightarrow \text{large}), |e'\rangle$.

Qualitative time-dependence

If m is varied "slowly" in time (e.g. $m = \alpha t$ with α small), then if the system starts in $|g\rangle$ with $E = E_1(m=0)$, ^{then} we expect to end up in $|e'\rangle$ with $E = E_2(m = \text{large})$.

Likewise, we expect $|e\rangle$ with $E = E_2(m=0) \rightarrow |g'\rangle$ with $E = E_1$ (m large)

IF m is varied "quickly" in time (e.g. $m = \alpha t$ with α large), then we expect the system to "jump" the no-level crossing gap:

$$E = E_1(m=0), |g\rangle \rightarrow E = E_1(m=\text{large}), |g'\rangle$$

$$E = E_2(m=0), |e\rangle \rightarrow E = E_2(m=\text{large}), |e'\rangle$$

Quantitative time-dependence

what is the probability to end up in state $|g'\rangle$ ($m \rightarrow +\infty$) given an initial state of $|g\rangle$ ($m \rightarrow -\infty$) and an energy ramp

$$\frac{d(E_2 - E_1)}{dt} = \frac{dE_{21}}{dt} = \gamma?$$

Answer: $P_{|g\rangle \rightarrow |g'\rangle} = \exp[-2\pi \Gamma]$

$|g\rangle \rightarrow |g'\rangle$ "jump" is called a Landau-Zener transition with $\Gamma = \frac{|W|^2}{\hbar \cdot \frac{dE_{21}}{dt}} = \frac{|W|^2}{\hbar \gamma} = \frac{[J]^2}{([J] \cdot [S])[J]} = \frac{[J]}{[S]}$
= dimensionless

Fast ramp $\rightarrow \gamma$ large $\rightarrow \Gamma$ small $\Rightarrow P_{|g\rangle \rightarrow |g'\rangle} \sim 1$

Slow ramp $\rightarrow \gamma \sim 0 \rightarrow \Gamma$ large $\Rightarrow P_{|g\rangle \rightarrow |g'\rangle} \sim 0$

