

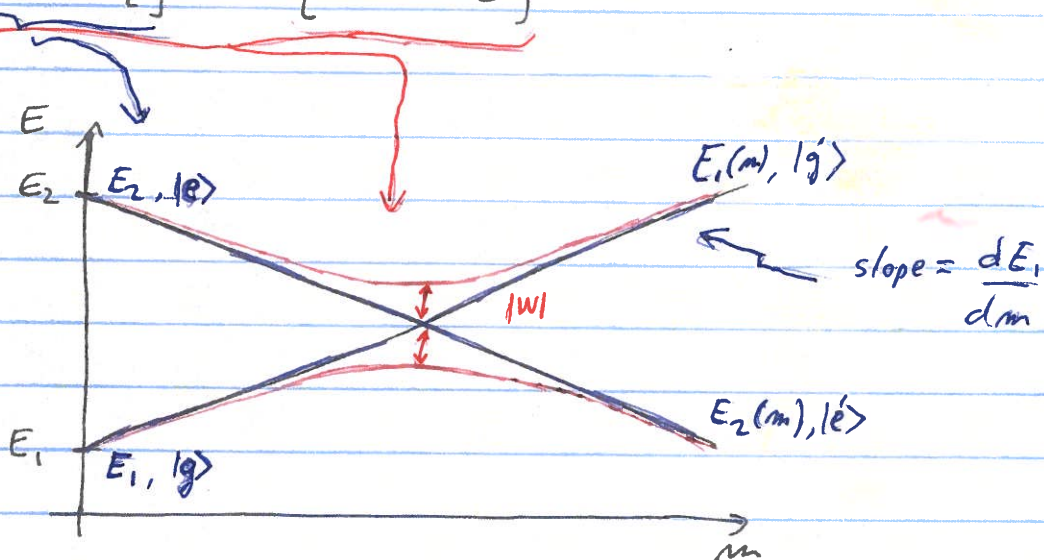
Tuesday, October 6, 2026

Landau - Zener Transitions

We consider a 2-level system with an off-diagonal interaction W (inter-state)

in which the base energies are varied through each other:

$$H = \begin{matrix} & \begin{matrix} |g\rangle & |e\rangle \end{matrix} \\ \begin{matrix} \langle g| \\ \langle e| \end{matrix} & \begin{bmatrix} E_1(m) & 0 \\ 0 & E_2(m) \end{bmatrix} \end{matrix} + \begin{bmatrix} 0 & W \\ W^* & 0 \end{bmatrix}$$



Effect of W

There is an avoided level crossing, and $E_2(m=0)$ is connected to $E_1(m \rightarrow \text{large}), |g'\rangle$, and $E_1(m=0, |g\rangle)$ is connected to $E_2(m \rightarrow \text{large}), |e'\rangle$.

Qualitative time-dependence

If m is varied "slowly" in time (e.g. $m = \alpha t$ with α small), then if the system starts in $|g\rangle$ with $E = E_1(m=0)$, ^{then} we expect to end up in $|e'\rangle$ with $E = E_2(m = \text{large})$.

Likewise, we expect $|e\rangle$ with $E = E_2(m=0) \rightarrow |g'\rangle$ with $E = E_1$ (m=large)
 $\swarrow$ adiabatically

If m is varied "quickly" in time (e.g. $m = \alpha t$ with α large), then we expect the system to "jump" the no-level crossing gap:

$$E = E_1(m=0), |g\rangle \rightarrow E = E_1(m=\text{large}), |g'\rangle$$

$$E = E_2(m=0), |e\rangle \rightarrow E = E_2(m=\text{large}), |e'\rangle$$

Quantitative time-dependence

what is the probability to end up in state $|g'\rangle$ (m large)
 given an initial state of $|g\rangle$ (m=0) and an energy ramp
 $m \rightarrow +\infty$
 $m \rightarrow -\infty$

$$\frac{d(E_2 - E_1)}{dt} = \frac{dE_{21}}{dt} = \gamma?$$

Answer:

$$P_{|g\rangle \rightarrow |g'\rangle} = \exp[-2\pi \Gamma]$$

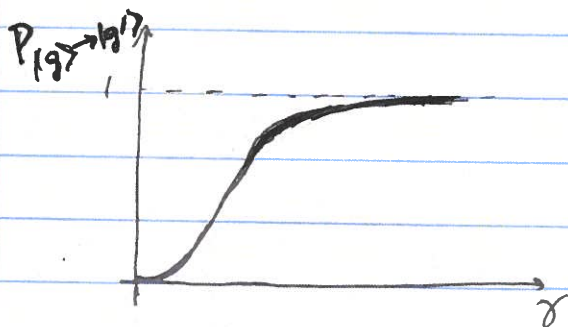
$|g\rangle \rightarrow |g'\rangle$ "jump"

is called a Landau-Zener transition

$$\text{with } \Gamma = \frac{|W|^2}{\hbar \cdot \frac{dE_{21}}{dt}} = \frac{|W|^2}{\hbar \gamma} \quad \frac{[J]^2}{([J][S])[J]} \quad [S] = \text{dimensionless}$$

Fast ramp $\rightarrow \gamma$ large $\rightarrow \Gamma$ small $\Rightarrow P_{|g\rangle \rightarrow |g'\rangle} \sim 1$

Slow ramp $\rightarrow \gamma \sim 0 \rightarrow \Gamma$ large $\Rightarrow P_{|g\rangle \rightarrow |g'\rangle} \sim 0$



Solving the 1D Schrodinger Equation (for a single particle)

$$H = \frac{P^2}{2m} + V(x)$$

$$\Rightarrow -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi_E(x) + V(x) \psi_E(x) = E \psi_E(x)$$

Equation is real, initial conditions are real

$\Rightarrow$ the solutions can be made real.

If $\psi_E(x)$ is a solution, then $\psi_E^*(x)$ is also a solution with the same eigen energy.

Example: In the free particle case ($V=0$), $\psi_E(x) = \begin{cases} e^{ikx} \\ e^{-ikx} \end{cases}$ with $E = \frac{\hbar^2 k^2}{2m}$

and $\psi_E(x) = \begin{cases} \cos(kx) \\ \sin(kx) \end{cases}$ are also solutions
real

Constant Potential: $V(x) = V_0$

$$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi_E(x) + V_0 \psi_E(x) = E \psi_E(x)$$

$$\Leftrightarrow -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi_E(x) = (E - V_0) \psi_E(x)$$

$$\Leftrightarrow \left[\frac{\partial^2}{\partial x^2} + \underbrace{\frac{2m(E - V_0)}{\hbar^2}}_{k^2} \right] \psi_E(x) = 0$$

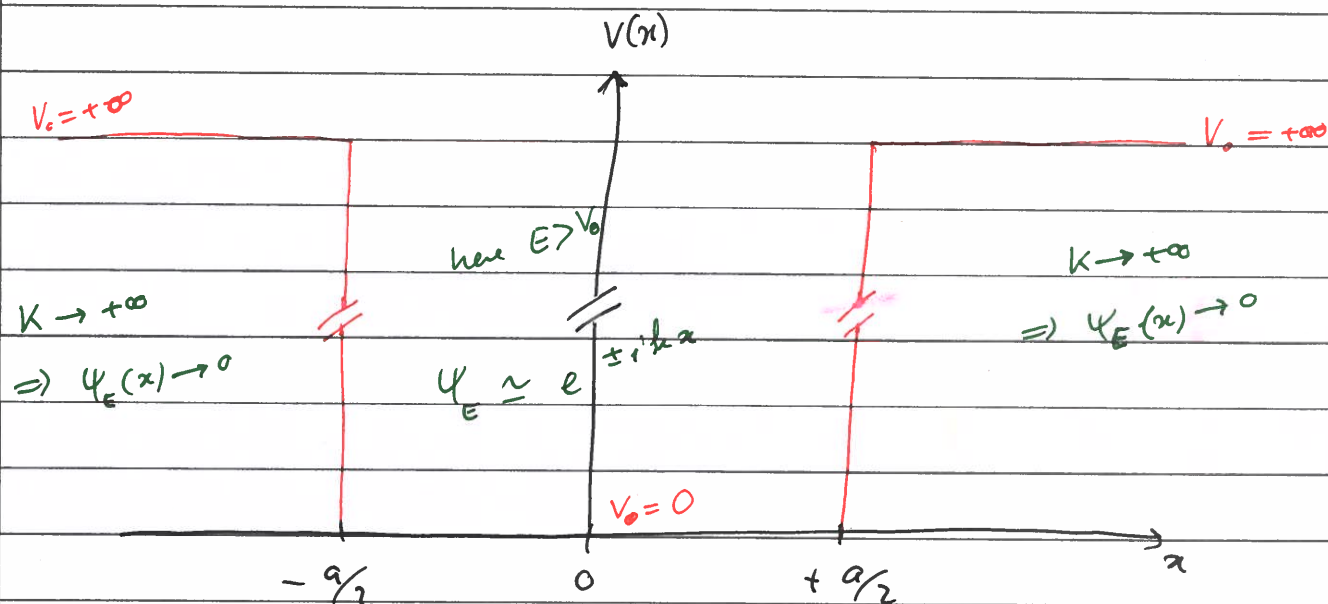
Helmholtz wave equation

$$k^2 = \frac{2m}{\hbar^2} (E - V_0) \quad (\Leftrightarrow) \quad E_k = \frac{\hbar^2 k^2}{2m} + V_0$$

Case 1: If $E > V_0$, then $k \in \mathbb{R}$ and $\psi_E(x) = \begin{cases} e^{ikx} \\ e^{-ikx} \end{cases}$
 $k = \sqrt{\frac{2m}{\hbar^2} (E - V_0)}$

Case 2: If $E < V_0$, then $k = \text{imaginary}$ and $\psi_E(x) = \begin{cases} e^{-Kx} \\ e^{+Kx} \end{cases}$
 $= \pm iK$
 $K = \sqrt{\frac{2m}{\hbar^2} (V_0 - E)}$

Infinite Square Well Potential

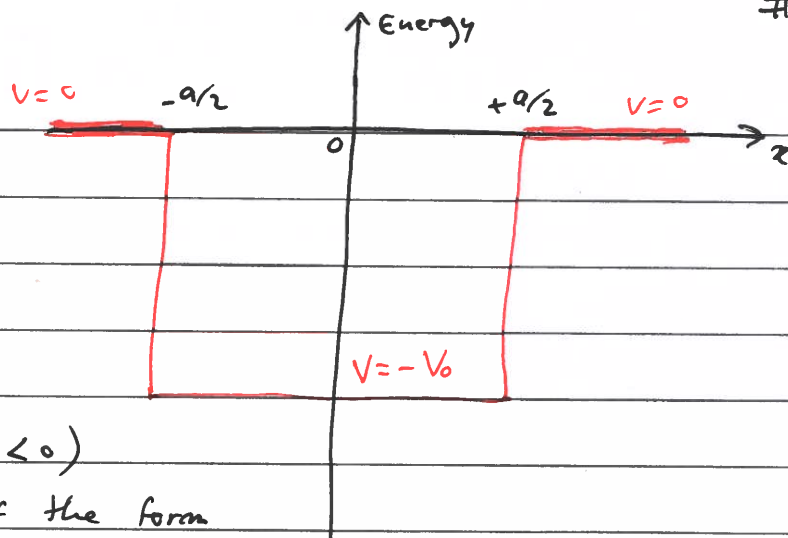


Since $\begin{cases} \psi_E(x \leq -a/2) = 0 \\ \psi_E(x \geq a/2) = 0 \end{cases}$ | we can use these requirements as boundary conditions for $\psi_E(-a/2 \leq x \leq a/2)$

$$\Rightarrow \psi_n = \begin{cases} \sqrt{\frac{2}{a}} \cos\left(n\pi \frac{x}{a}\right) & n = \text{odd} \\ \sqrt{\frac{2}{a}} \sin\left(n\pi \frac{x}{a}\right) & n = \text{even} \end{cases} \quad \left| \quad \text{with } E_n = \frac{\hbar^2 \pi^2 n^2}{2ma^2} \right.$$

$\Rightarrow$ quantized energy spectrum

$n = 0, 1, 2, \dots, \infty$

Finite Square WellBound state solutions ($E < 0$)

The solutions must be of the form

$$\left\{ \begin{array}{l} \psi(x \leq -a/2) = A_- e^{Kx} \\ \psi(x \geq +a/2) = A_+ e^{-Kx} \end{array} \right.$$

exponential decay in the classically "forbidden" regions

$$\left\{ \psi\left(-\frac{a}{2} \leq x \leq +\frac{a}{2}\right) = B_+ \cos(kx) + B_- \sin(kx) \right. \quad \left. \begin{array}{l} \text{standing wave in} \\ \text{the classically "allowed"} \\ \text{region.} \end{array} \right.$$

with $K = \sqrt{\frac{2m}{\hbar^2} |E|}$

$$k = \sqrt{\frac{2m}{\hbar^2} (V_0 - |E|)}$$

note: the bound states have $E < 0$

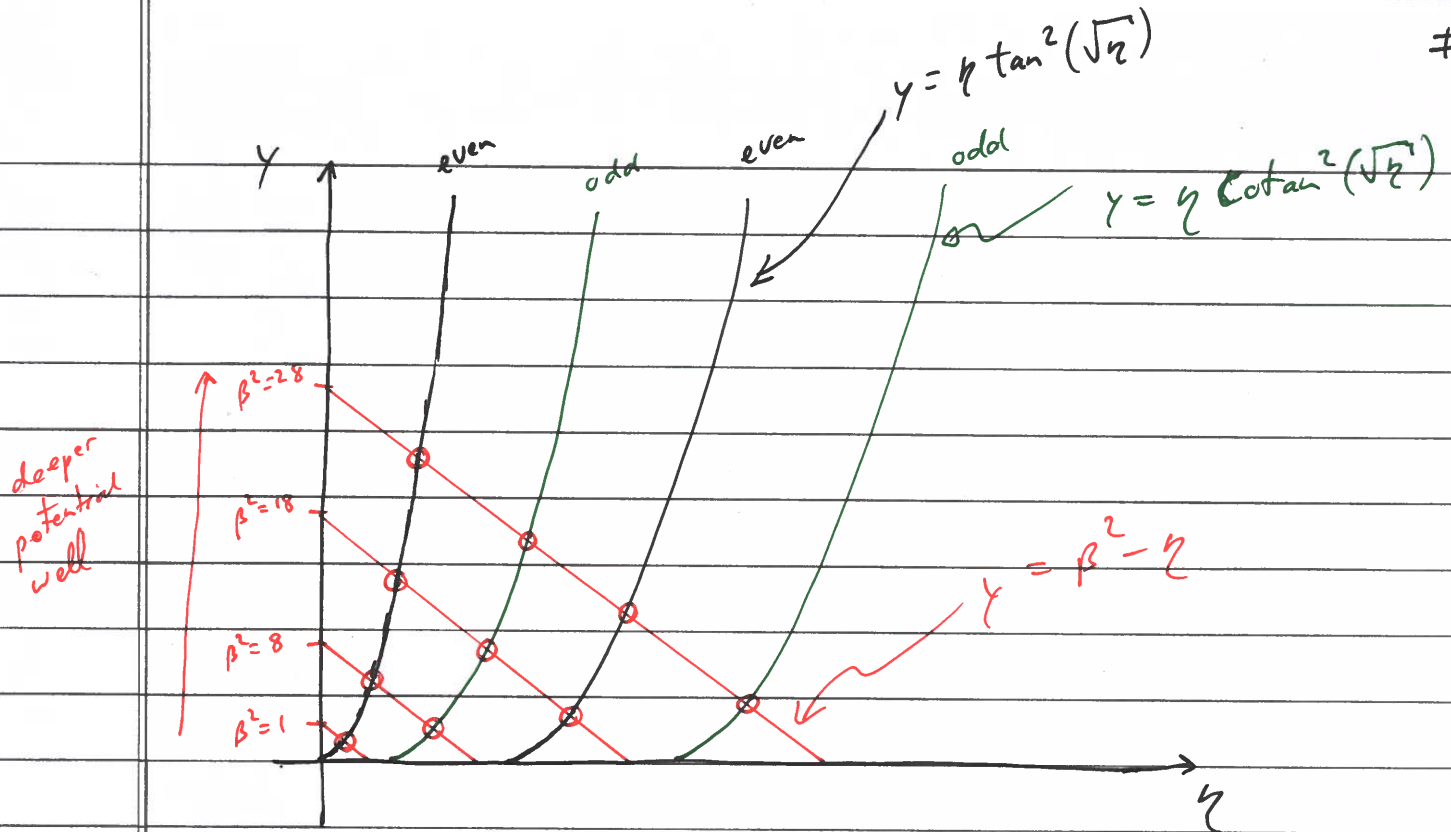
Boundary conditions: We require $\psi(x)$ and $\frac{\partial \psi(x)}{\partial x}$ to be continuous (notably at $x = \pm a/2$)

↳ require continuity of probability current $J(x) = \psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}$

Define
$$\left\{ \begin{array}{l} k^2 = \frac{ma^2}{2\hbar^2} (V_0 - |E|) \\ \beta^2 = \frac{ma^2}{2\hbar^2} \cdot V_0 \end{array} \right.$$

odd symmetry solutions: $\beta^2 - k^2 = \gamma \cot^2(\sqrt{\gamma})$ for $\tan(\sqrt{\gamma}) \leq 0$

even symmetry solutions: $\beta^2 - k^2 = \gamma \tan^2(\sqrt{\gamma})$ for $\tan(\sqrt{\gamma}) \geq 0$



⇒ use intersections to get eigenenergies

Note: There is a continuous spectrum of eigenenergies for $E > 0$ eigenstates

Harmonic Oscillator Potential: $V = \frac{1}{2} m \omega^2 x^2$

Very important model

↳ most cold atom traps are harmonic

↳ The EM field can be described as an abstract harmonic oscillator.

$$H = \frac{1}{2m} P^2 + \frac{1}{2} m \omega^2 X^2$$

We define the annihilation and creation operators:

$$a = \sqrt{\frac{m\omega}{2\hbar}} \left(X + i \frac{1}{m\omega} P \right) \text{ and } a^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left(X - i \frac{1}{m\omega} P \right)$$