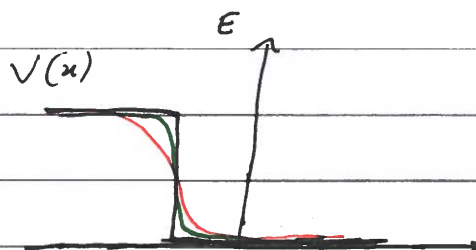


Tuesday, October 13, 2026

Continuity of the wavefunction $\psi(x)$

Not ^{necessarily} required by continuity of $J(x) = \psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}$
as incorrectly stated in previous lecture

- 1) Discontinuous potentials should be viewed as the limit of a continuous potential:



Continuous $V(x) \rightarrow$ Continuous $\psi(x)$

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x)}{\partial x^2} + V(x) \psi(x) = E \psi(x)$$

Discontinuous $\psi(x) \Rightarrow \begin{cases} \frac{\partial \psi}{\partial x} \\ \frac{\partial^2 \psi}{\partial x^2} \end{cases}$ is ill-defined or infinite

- 2) First derivative of $\psi(x)$ (i.e. $\frac{\partial \psi}{\partial x}$) should be continuous at a boundary $x = x_0$:

integrate

$$\frac{\partial^2 \psi(x)}{\partial x^2} = \frac{2m}{\hbar^2} (V(x) - E) \psi(x)$$

$$\Rightarrow \int_{x_0 - \epsilon}^{x_0 + \epsilon} \left[\frac{\partial^2 \psi(x)}{\partial x^2} \right] dx = \frac{2m}{\hbar^2} \int_{x_0 - \epsilon}^{x_0 + \epsilon} (V(x) - E) \psi(x) dx$$

$\frac{\partial \psi(x)}{\partial x} \Big|_{x_0 - \epsilon}^{x_0 + \epsilon}$
tends $\rightarrow 0$ for $\epsilon \rightarrow 0$
even if $V(x)$ is discontinuous

[or $\psi(x)$ is ~~discontinuous~~ discontinuous]

$$\Rightarrow \frac{\partial \psi}{\partial x} \Big|_{x=x_0+\epsilon} = \frac{\partial \psi}{\partial x} \Big|_{x=x_0-\epsilon} \Rightarrow \boxed{\frac{\partial \psi}{\partial x} \text{ is continuous}}^*$$

Conclusion: $\psi(x)$ and $\frac{\partial \psi}{\partial x}$ are continuous

* unless $V(x)$ includes a Dirac Delta.

Harmonic Oscillator Potential : $V(x) = \frac{1}{2} m \omega^2 x^2$

$\omega = \text{classical oscillation frequency}$

The Hamiltonian for a particle is then

$$H = \frac{1}{2m} P^2 + \frac{1}{2} m \omega^2 X^2$$

so the Schrodinger equation is thus

$$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi(x) + \frac{1}{2} m \omega^2 x^2 \psi(x) = E \psi(x)$$

The solution involves Hermite functions:

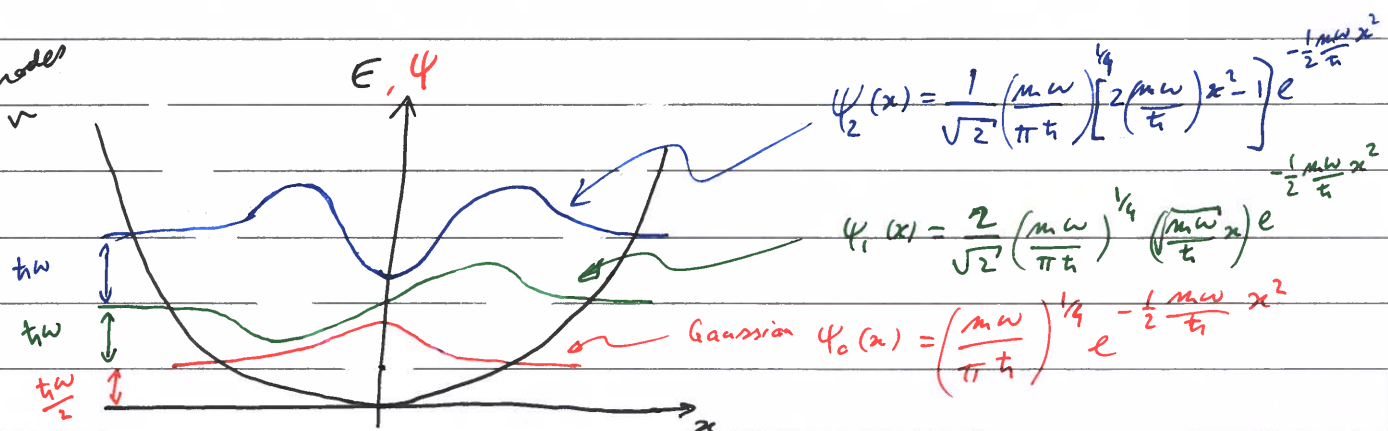
Eigenstates: $\psi_n(x) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi \hbar} \right)^{1/4} H_n \left(\sqrt{\frac{m\omega}{\hbar}} x \right) e^{-\frac{1}{2} \frac{m\omega}{\hbar} x^2}$ (orthonormal basis)

Eigenenergies: $E_n = \hbar \omega \left(n + \frac{1}{2} \right)$ where $n = 0, 1, 2, \dots, \infty$

where $H_n(z)$ is Hermite polynomial of degree n .

$$H_0(z) = 1, \quad H_1(z) = 2z, \quad H_2(z) = 4z^2 - 2, \quad H_3(z) = 8z^3 - 12z$$

note: # of nodes = n



Creation & Annihilation operators

We can define the operators (non-Hermitian!):

annihilation: $a = \sqrt{\frac{m\omega}{2\hbar}} \left(X + i \frac{1}{m\omega} P \right)$ creation: $a^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left(X - i \frac{1}{m\omega} P \right)$

with $[a, a^\dagger] = 1$

The Hamiltonian then becomes $H = \hbar\omega \left(a^\dagger a + \frac{1}{2} \right)$

We can define the number operator $N = a^\dagger a$

$\Rightarrow H = \hbar\omega \left(N + \frac{1}{2} \right)$

Note: N is Hermitian, since $N^\dagger = (a^\dagger a)^\dagger = (a)^\dagger (a^\dagger)^\dagger = a^\dagger a = N$
[recall $(uv)^\dagger = v^\dagger u^\dagger$]

sometimes called "Fock states" in quantum optics

the eigenstates for N are the number states $|n\rangle$:

orthonormal basis

$N|n\rangle = n|n\rangle$ where $n = 0, 1, 2, \dots, \infty$

$H|n\rangle = \hbar\omega \left(n + \frac{1}{2} \right) |n\rangle$ so the eigenenergies of H are $E_n = \hbar\omega \left(n + \frac{1}{2} \right)$

$|n\rangle$ are the eigenstates of H , so we have

$\langle x|n\rangle = \psi_n(x) = \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} H_n \left(\sqrt{\frac{m\omega}{\hbar}} x \right) e^{-\frac{1}{2} \frac{m\omega}{\hbar} x^2}$

Action of a and $a^\dagger$ on $|n\rangle$:

$$a|n\rangle = \sqrt{n} |n-1\rangle, \quad a^\dagger|n\rangle = \sqrt{n+1} |n+1\rangle$$

Average & Variance:

$$\langle N \rangle_n = \langle n|N|n\rangle = \langle n|n|n\rangle = n$$

$$\langle N^2 \rangle_n = \langle n|N^2|n\rangle = \langle n|Nn|n\rangle = \langle n|n^2|n\rangle = n^2$$

$$\langle \Delta N^2 \rangle_n = \langle n|(N - \langle N \rangle_n)^2|n\rangle = n^2 - n^2 = 0$$

The average excitation of state $|n\rangle$ is n with zero variance.

Coherent states:

Classical ^{state} motion of a particle in a harmonic oscillator potential is represented by a coherent state $|\alpha\rangle$: (α = complex number)

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \quad \leftarrow \text{normalized}$$

$\langle \alpha|\alpha\rangle = 1$

A coherent state is an eigenstate of the annihilation operator:

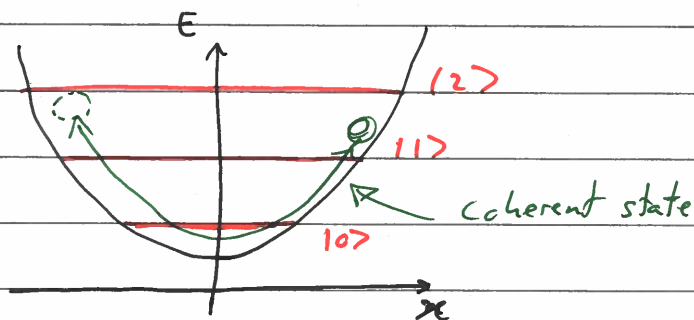
$$a|\alpha\rangle = \alpha|\alpha\rangle$$

$$\langle \alpha|a^\dagger = \langle \alpha|\alpha^*$$

Coherent states are generally not orthogonal to each other
i.e., $\langle \alpha|\beta\rangle \neq 0$

actually, $|\langle \alpha | \beta \rangle|^2 = \exp(-|\alpha - \beta|^2)$

Intuitive picture:



Coherent states obey the minimum Heisenberg uncertainty relation:

$$\Delta X \cdot \Delta P = \frac{\hbar}{2}$$

Average & variance:

$$\begin{aligned} \langle N \rangle_\alpha &= \langle \alpha | N | \alpha \rangle = \langle \alpha | a^\dagger a | \alpha \rangle = \langle \alpha | a^\dagger | \alpha | \alpha \rangle \\ &= \langle \alpha | | \alpha |^2 | \alpha \rangle \\ &= | \alpha |^2 \underbrace{\langle \alpha | \alpha \rangle}_{=1} \\ &= | \alpha |^2 \end{aligned}$$

$$\Rightarrow \langle N \rangle_\alpha = | \alpha |^2$$

$$\begin{aligned} \langle N^2 \rangle_\alpha &= \langle \alpha | N^2 | \alpha \rangle = \langle \alpha | \underbrace{a^\dagger a a^\dagger a}_{1 + a^\dagger a} | \alpha \rangle \\ &= \langle \alpha | a^\dagger a | \alpha \rangle + \langle \alpha | a^\dagger a^\dagger a a | \alpha \rangle \\ &= | \alpha |^2 + | \alpha |^4 \end{aligned}$$

$$\Rightarrow \langle \Delta N^2 \rangle_\alpha = \langle N^2 \rangle_\alpha - (\langle N \rangle_\alpha)^2$$

$$= \cancel{| \alpha |^4} + | \alpha |^2 - (\cancel{| \alpha |^2})^2 = | \alpha |^2 = \langle N \rangle_\alpha$$

thus $\langle \Delta N^2 \rangle_\alpha = \langle N \rangle_\alpha \Rightarrow \Delta N_\alpha = \sqrt{\langle \Delta N^2 \rangle_\alpha}$

$\Rightarrow \Delta N_\alpha = \sqrt{\langle N \rangle_\alpha}$

Summary: the spread ΔN of a coherent state is the square root of the average excitation $\langle N \rangle$, i.e. $\Delta N = \sqrt{\langle N \rangle}$

Reminder: If you are counting photons from a classical light source, and on average you get $\langle n \rangle$ photons per second, then in later seconds you can expect to get $\langle n \rangle \pm \sqrt{\langle n \rangle}$ photons. (67% of the time).

Displacement Operator:

Coherent states can also be defined (generated) using the "displacement operator": $D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}$

Diagram illustrating the displacement operator $D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}$. The expression is enclosed in a dashed blue box. Arrows point from the labels "scalars" to α and α^* , and from the label "operator" to $a^\dagger$ and a .

$D(\alpha)$ is unitary with $D^\dagger(\alpha) = e^{\alpha^* a - \alpha a^\dagger}$ and $D(\alpha) D^\dagger(\alpha) = 1$

Importantly, $D(\alpha) |0\rangle = |\alpha\rangle = e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle$