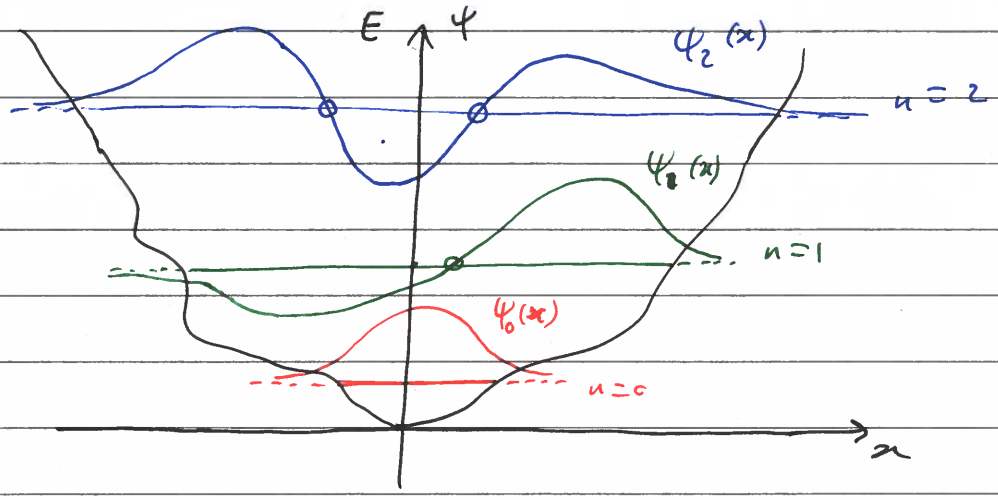
Thursday, October 15, 2026

Nodal theorem of quantum mechanics

In a 1D potential with bound state, the n^{th} bound eigenstate has n nodes (the ground state is labeled $n=0$).



node = position x_0 at which $\psi(x_0) = 0$

Side Note: In 1D, a wavefunction $\psi(x)$ cannot have a point x_0 such that $\psi(x_0) = 0$ and $\frac{\partial}{\partial x}\psi(x)\Big|_{x=x_0} = 0$.

"No double null" theorem [Named by S. Aubin!]

In other words, if there exists a point x_0 such that $\psi(x_0) = 0$ and $\frac{\partial}{\partial x}\psi(x)\Big|_{x=x_0} = 0$, then $\psi(x) = 0$ for all x.

[for well behaved $V(x)$ i.e. not infinite]

proof 1: The Schrodinger equation $-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x) + V(x)\psi(x) = E\psi(x)$

is a 2nd order differential equation $\psi''(x) + \underbrace{\frac{2m}{\hbar^2}(E - V(x))}_{= f(x)}\psi(x) = 0$

$\Leftrightarrow \quad \psi''(x) + f(x)\psi(x) = 0 \quad (1)$

with boundary conditions $\psi(x_0)=0$, $\psi'(x_0)=0$ and it admits a unique solution. (Diff. Eq. Theorem)

clearly $\psi(x)=0$ is a solution to eq. 1, and so it is the solution.

Proof 2: If $\psi(x_0)=0$, then by eq. 1 $\psi''(x_0)+f(x_0)\underbrace{\psi(x_0)}_{=0}=0$

$\Rightarrow \psi''(x_0)=0$

Also, $\frac{\partial}{\partial x}\left[\psi''(x)+f(x)\psi(x)=0\right] \Rightarrow \psi'''(x)+f'(x)\psi(x)+f(x)\psi'(x)=0$

at $x_0=0$, we have $\psi'''(x_0)+f'(x_0)\underbrace{\psi(x_0)}_{=0}+f(x_0)\underbrace{\psi'(x_0)}_{=0}=0$

$\Rightarrow \psi'''(x_0)$

If we keep differentiating eq. 1, then clearly all derivatives of $\psi(x)$ at $x=x_0$ are zero.

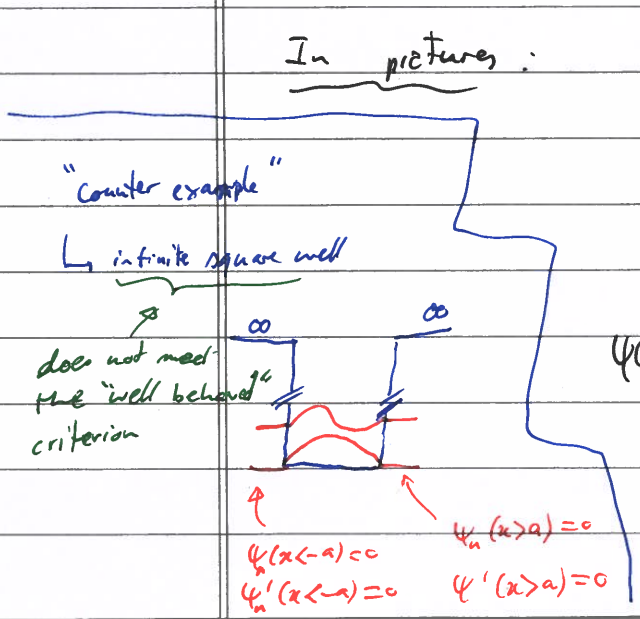
Next, we can Taylor expand $\psi(x)$ around $x=x_0$

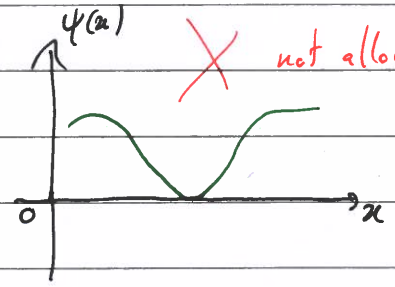
$\Rightarrow \psi(x)=\underbrace{\psi(x_0)}_{=0}+\frac{1}{1!}\underbrace{\psi'(x_0)}_{=0}(x-x_0)+\frac{1}{2!}\underbrace{\psi''(x_0)}_{=0}(x-x_0)^2+\cdots=0$

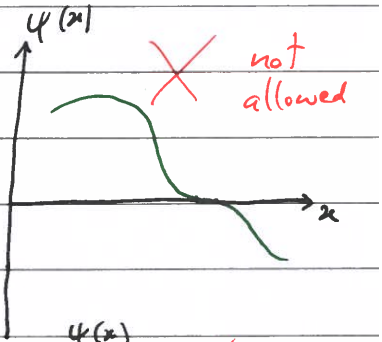
$=0$ over some interval of x

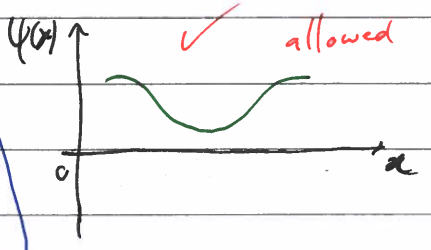
$\Rightarrow$ expand interval by Taylor expanding around ~~every~~ x close x_0 (inside interval)

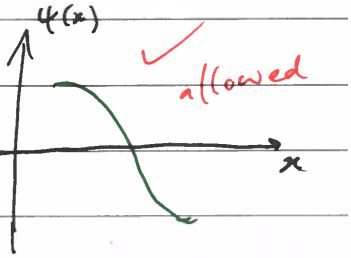
In pictures:











Back to the nodal theorem ... First, we need the Wronskian theorem:

Consider $\begin{cases} \psi_1''(x) + f_1(x)\psi_1(x) = 0 \\ \psi_2''(x) + f_2(x)\psi_2(x) = 0 \end{cases}$ | $f_1(x)$ and $f_2(x)$ are piecewise continuous on an interval $x \in (a,b)$

The Wronskian $W(\psi_1, \psi_2)$ is given by:

$$W(\psi_1, \psi_2) = \psi_1(x)\psi_2'(x) - \psi_2(x)\psi_1'(x)$$

Next, we note that $\frac{\partial}{\partial x} W(\psi_1, \psi_2) = \psi_1'(x)\psi_2'(x) + \psi_1(x)\psi_2''(x) - \psi_2'(x)\psi_1'(x) - \psi_2(x)\psi_1''(x)$

($-f_2(x)\psi_2(x)$; $-f_1(x)\psi_1(x)$)

$$\Rightarrow \frac{\partial}{\partial x}\left[\psi_1(x)\psi_2'(x) - \psi_2(x)\psi_1'(x)\right] = \left(f_1(x) - f_2(x)\right)\psi_1(x)\psi_2(x)$$

integrate

$$\Rightarrow \int_a^b dx \frac{\partial}{\partial x}\left[\psi_1\psi_2' - \psi_2\psi_1'\right] = \int_a^b dx \left[\left(f_1(x) - f_2(x)\right)\psi_1(x)\psi_2(x)\right]$$

$$\boxed{\left[\psi_1(x)\psi_2'(x) - \psi_2(x)\psi_1'(x)\right]_a^b = \int_a^b \left(f_1(x) - f_2(x)\right)\psi_1(x)\psi_2(x)\,dx}$$

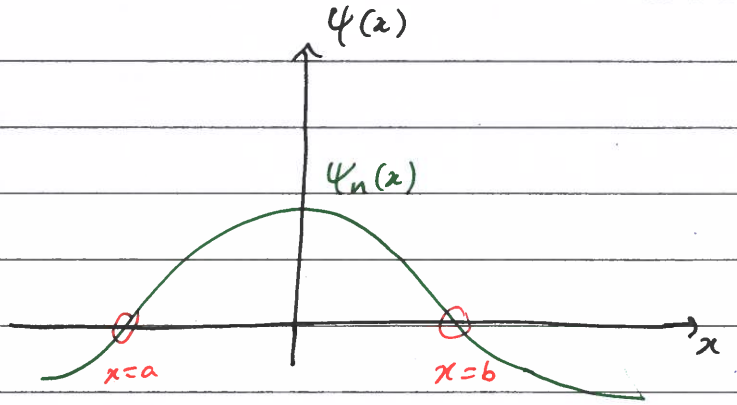
Wronskian theorem

Back to the nodal theorem:

Consider two adjacent/neighboring nodes at $x = a$ & $x = b$ for $\psi_n(x)$.

Let $\begin{cases} \psi_n(x) \text{ be the eigenstate} \\ E_n \text{ be the eigenenergy} \end{cases}$ of the n^{th} bound state

Let $\begin{pmatrix} \psi_{n+1}(x) \\ E_{n+1} \end{pmatrix}$ be the $\begin{pmatrix} \text{eigenstate} \\ \text{eigenenergy} \end{pmatrix}$ of the $n+1^{th}$ bound state



Thus $\psi_n''(x) + \underbrace{\frac{2m}{\hbar^2}\left(E_n - V(x)\right)}_{f_n(x)} \psi_n(x) = 0$

$$\psi_{n+1}''(x) + \underbrace{\frac{2m}{\hbar^2}\left(E_{n+1} - V(x)\right)}_{f_{n+1}(x)} \psi_{n+1}(x) = 0$$

According to the Wronskian theorem:

$$\left[\psi_{n+1}(x)\,\psi_n'(x) - \psi_n(x)\,\psi_{n+1}'(x)\right]_a^b = \int_a^b \underbrace{\left(f_{n+1}(x) - f_n(x)\right)}_{\frac{2m}{\hbar^2}\left(E_{n+1} - E_n\right)} \psi_n(x)\psi_{n+1}(x)\,dx$$

$$= \underbrace{\frac{2m}{\hbar^2}\left(E_{n+1} - E_n\right)}_{>0 \text{ since } E_{n+1} > E_n} \int_a^b \psi_n(x)\,\psi_{n+1}(x)\,dx$$

eq. 2

Suppose $\psi_n(a<x<b) > 0$, then $\begin{cases} \psi_n'(a) > 0 \\ \psi_n'(b) < 0 \end{cases}$

proof by contradiction

Next, assume $\psi_{n+1}(x)$ has no nodes between $a<x<b$.

For convenience, let $\psi_{n+1}(a<x<b) > 0$ (positive)

↳ $\Rightarrow$ RHS of eq. 2 is positive. $\left[\frac{2m}{\hbar^2}(E_{n+1}-E_n)\int_a^b \psi_n \psi_{n+1}\,dx\right] > 0$

Let's look at the LHS:

$$\text{LHS} = \underbrace{\psi_{n+1}(b)}_{>0}\underbrace{\psi_n'(b)}_{<0} - \cancel{\psi_n(b)\psi_{n+1}'(b)}^{\,0} - \Big[\underbrace{\psi_{n+1}(a)}_{>0}\underbrace{\psi_n'(a)}_{>0} - \cancel{\psi_n(a)\psi_{n+1}'(a)}^{\,0}\Big]$$

$\underbrace{\qquad\qquad}_{<0}$ $\qquad$ $\underbrace{\qquad\qquad}_{>0}$

$=$ negative

Clearly, LHS $\neq$ RHS $\Rightarrow$ Assumption that $\psi_{n+1}(x)$ has no nodes between $a<x<b$ is incorrect.

$\Rightarrow$ ψ_{n+1} has at least one node between any pair of nodes of ψ_n.

ψ_{n+1} has at least one more node than ψ_n.

bound eigenstate

In general, ψ_n has has exactly n nodes

Intuitive proof

Consider the infinite square well potential or the harmonic oscillator potential which both have this property.

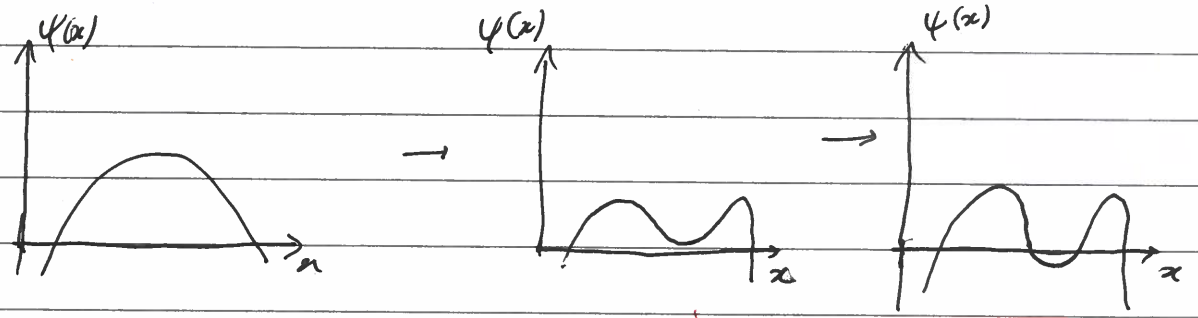
Now deform either one of these potentials into your desired potential. As you perform the deformation in a continuous manner, then the eigenstates will also be modified, but their number of nodes will remain unchanged.

Important: To introduce a new node would require the appearance of a point x_0 where $\begin{cases} \psi(x_0) = 0 \\ \text{and} \\ \psi'(x_0) = 0 \end{cases}$

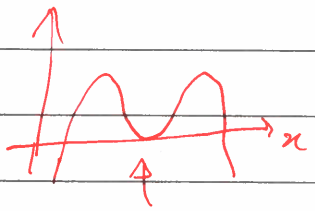
which is not permitted see "side note"

"No double null" theorem

Example:



not permitted because



$\psi = 0, \psi' = 0$ is not permitted