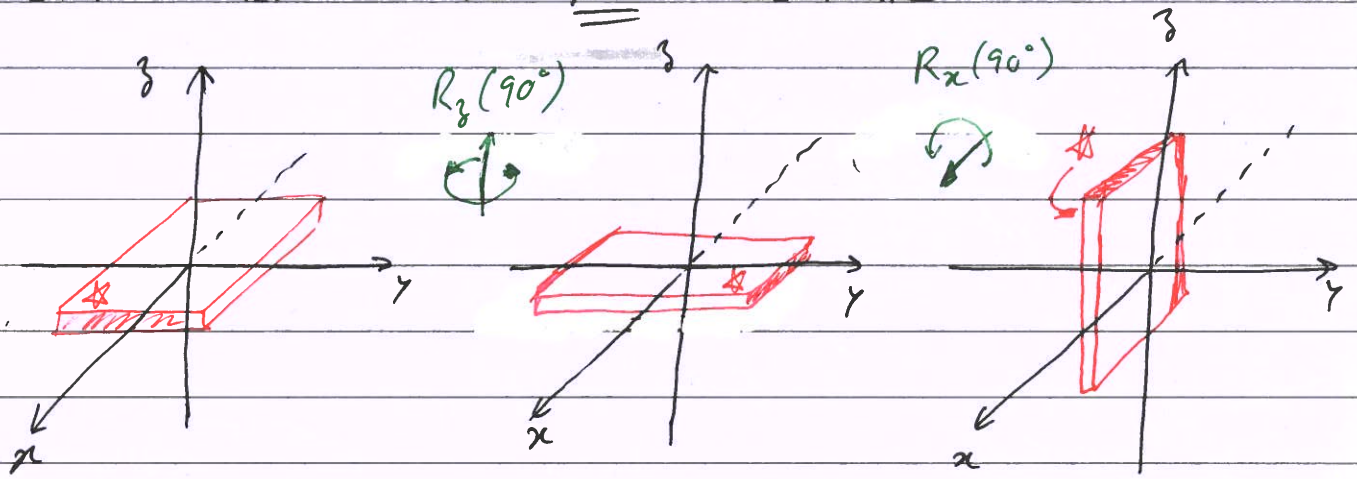
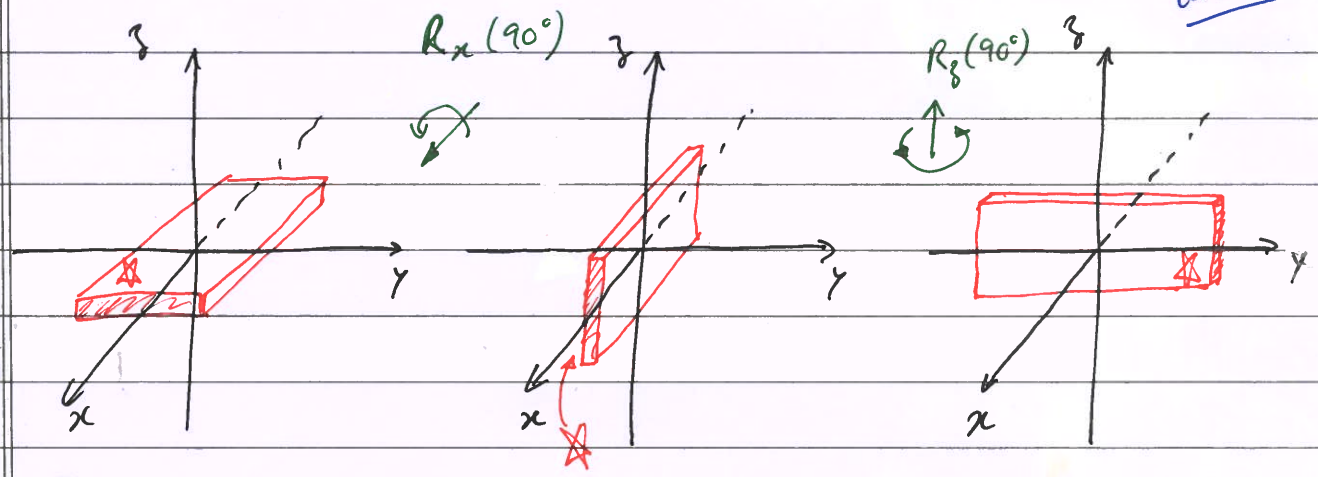
Tuesday, October 20, 2026

Rotations & Angular Momentum (chpt 3.1)

Classical Rotations do NOT Commute



these orientations are very different



Clearly $R_x(90°)\, R_z(90°)\, \underbrace{\vec{v}}_{\hat{x}} \neq R_z(90°)\, R_x(90°)\, \underbrace{\vec{v}}_{\hat{x}}$

$= \hat{z}$ $\qquad$ $= \hat{y}$

The rotation of a vector $\vec{v}$ in 3D can be performed via a matrix / operator $R_{\hat{n}}(\theta)$:

$$\vec{v}\,' = R_{\hat{n}}(\theta)\,\vec{v}$$

note: R is unitary & real, i.e. "orthogonal", so $R^T R = 1$

Rotations by a counterclockwise angle θ about the cartesian axes are given by

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \quad R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In the limit of very small rotations $\theta \to \varepsilon$, we get (drop terms of order ε^2 and higher)

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\varepsilon \\ 0 & \varepsilon & 1 \end{pmatrix}$$

$$R_y(\theta) = \begin{pmatrix} 1 & 0 & \varepsilon \\ 0 & 1 & 0 \\ -\varepsilon & 0 & 1 \end{pmatrix}, \quad R_z(\theta) = \begin{pmatrix} 1 & -\varepsilon & 0 \\ \varepsilon & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Next, we note that

$$R_x(\varepsilon)\,R_y(\varepsilon) = \begin{pmatrix} 1 & 0 & \varepsilon \\ \varepsilon^2 & 1 & -\varepsilon \\ -\varepsilon & \varepsilon & 1 \end{pmatrix}$$

$$R_y(\varepsilon)\,R_x(\varepsilon) = \begin{pmatrix} 1 & \varepsilon^2 & \varepsilon \\ 0 & 1 & -\varepsilon \\ -\varepsilon & \varepsilon & 1 \end{pmatrix}$$

$$\Rightarrow \quad R_x(\varepsilon)\,R_y(\varepsilon) - R_y(\varepsilon)\,R_x(\varepsilon) = \begin{pmatrix} 0 & -\varepsilon^2 & 0 \\ \varepsilon^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = R_z(\varepsilon^2) - 1$$

We note that $R_z(\varepsilon) = 1 - i\,\varepsilon\,J_z$ with $J_z = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$R_x(\varepsilon) = 1 - i\,\varepsilon\,J_x \quad \text{with} \quad J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$R_y(\varepsilon) = 1 - i\,\varepsilon\,J_y \quad \text{with} \quad J_y = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}$$

$J_{x,y,z}$ are "generators" of rotations, since we can now construct finite rotations with them

$$R_{\substack{x\\y\\z}}(\Theta) = \lim_{N\to\infty}\left(1 - i\left(\frac{\Theta}{N}\right)J_{\substack{x\\y\\z}}\right)^N = e^{-i\,\Theta\,J_{\substack{x\\y\\z}}}$$

We also note that the commutation relations obey

$$\left[\mathcal{J}_x, \mathcal{J}_y\right] = i\,\mathcal{J}_z$$

and $$\left[\mathcal{J}_i, \mathcal{J}_j\right] = i\,\epsilon_{ijk}\,\mathcal{J}_k$$

ϵ_{ijk} = Levi-Civita symbol → completely anti-symmetric under exchange of indices

Example: $\epsilon_{123} = 1$, $\epsilon_{213} = -1$, $\epsilon_{231} = 1$, $\epsilon_{321} = -1$, $\epsilon_{312} = 1$, ...

$\epsilon_{113} = 0$ ← always zero for repeated indices

Angular Momentum:

Orbital angular momentum: $\vec{L} = \vec{R} \times \vec{P}$ $\quad (R_1 = X, R_2 = Y, R_3 = Z)$

We note that $L_i = \sum_{j,k=1}^{3} \epsilon_{ijk}\, R_j\, P_k$

Example: $L_z = L_3 = \underbrace{\epsilon_{312}}_{=+1} R_1 P_2 + \underbrace{\epsilon_{321}}_{=-1} R_2 P_1$

$$= R_1 P_2 - R_2 P_1 = X P_y - Y P_x$$

$$L_x = L_1 = \underbrace{\epsilon_{123}}_{=+1} R_2 P_3 + \underbrace{\epsilon_{132}}_{=-1} R_3 P_2 = R_2 P_3 - R_3 P_2$$

$$= Y P_z - Z P_y$$

$$L_y = L_2 = \underbrace{\epsilon_{231}}_{=+1} R_3 P_1 + \underbrace{\epsilon_{213}}_{=-1} R_1 P_3 = R_3 P_1 - R_1 P_3$$

$$= Z P_x - X P_z$$

Commutators:

$$[L_x, L_y] = [YP_z - ZP_y \,,\, ZP_x - XP_z]$$

since $[YP_z, -XP_z] = 0$ and $[-ZP_y, ZP_x] = 0$

$$= [YP_z, ZP_x] + [ZP_y, XP_z]$$

$$= YP_x \underbrace{[P_z, Z]}_{-i\hbar\mathbb{1}} + XP_y \underbrace{[Z, P_z]}_{i\hbar\mathbb{1}}$$

$$= i\hbar \underbrace{(XP_y - YP_x)}_{L_z}$$

$$\Rightarrow \boxed{[L_x, L_y] = i\hbar L_z}$$

Similarly, $\boxed{[L_y, L_z] = i\hbar L_x}$ and $\boxed{[L_z, L_x] = i\hbar L_y}$

more generally: $\boxed{[L_i, L_j] = i\hbar \epsilon_{ijk} L_k}$ $\quad i,j,k = \begin{cases} 1,2,3 \\ x,y,z \end{cases}$

Generalized Angular Momentum

We define an angular momentum $\vec{J}$ as any set of observables J_x, J_y, J_z which satisfies:

$$\boxed{[J_i, J_j] = i\hbar \epsilon_{ijk} J_k}$$

also $J_{i,j,k}$ are Hermitian

This angular momentum can serve as a generator of rotations:

note: $\hbar$ has units of angular momentum

$$\boxed{R_{\hat{n}}(d\phi) = 1 - \frac{i}{\hbar}(\vec{J}\cdot\hat{n})\,d\phi}$$

$$\Rightarrow \boxed{R_{\hat{n}}(\phi) = \lim_{N\to\infty}\left(1 - \frac{i}{\hbar}(\vec{J}\cdot\hat{n})\frac{\phi}{N}\right)^N = e^{-i\frac{\phi}{\hbar}(\vec{J}\cdot\hat{n})}}$$

Next, we consider $\boxed{\vec{J}^2 = J_x^2 + J_y^2 + J_z^2}$

We can calculate $[J^2, J_i]$:

$$[J^2, J_x] = [J_x^2 + J_y^2 + J_z^2, J_x] = [J_y^2, J_x] + [J_z^2, J_x]$$

$$= J_y[J_y, J_x] + [J_y, J_x]J_y + J_z[J_z, J_x] + [J_z, J_x]J_z$$

($[J_y, J_x] = -i\hbar J_z$, $[J_y, J_x] = -i\hbar J_z$, $[J_z, J_x] = i\hbar J_y$, $[J_z, J_x] = i\hbar J_y$)

Commutator Identity: $[AB, C] = A[B, C] + [A, C]B$

$$= -i\hbar J_y J_z - i\hbar J_z J_y + i\hbar J_z J_y + i\hbar J_y J_z$$

$$= 0$$

more generally $\Rightarrow$ $\boxed{[\vec{J}^2, \vec{J}] = 0}$

$\Rightarrow$ We can obtain simultaneous eigenstates for both $\vec{J}^2$ and J_x or J_y or J_z

Simultaneous eigenstates of $\vec{J}^2$ and J_z (and their eigenvalues)

$$J^2|a, b\rangle = a|a, b\rangle$$
$$J_z|a, b\rangle = b|a, b\rangle$$

We are interested in determining a and b.

Define: $\boxed{J_\pm = J_x \pm iJ_y}$ ("Raising" & "Lowering" ladder operators)

Then $[J_z, J_\pm] = [J_z, J_x \pm iJ_y] = [J_z, J_x] \pm i[J_z, J_y]$

($[J_z, J_x] = i\hbar J_y$, $[J_z, J_y] = -i\hbar J_x$)

$$= \pm\hbar J_x + i\hbar J_y = \pm\hbar(J_x \pm iJ_y)$$

($J_x \pm iJ_y = J_\pm$)

$$\Rightarrow \boxed{[J_z, J_\pm] = \pm\hbar J_\pm}$$

Also $\boxed{[\vec{J}^2, J_\pm] = 0}$

Furthermore, $[J_+, J_-] = [J_x + iJ_y, J_x - iJ_y]$

$= -i[J_x, J_y] + i[J_y, J_x] = \hbar J_z + \hbar J_z$

($[J_x, J_y] = i\hbar J_z$, $[J_y, J_x] = -i\hbar J_z$)

$$\Rightarrow \boxed{[J_+, J_-] = 2\hbar J_z}$$

Since $[J_z, J_\pm] = \pm\hbar J_\pm \Leftrightarrow J_z J_\pm = J_\pm J_z \pm \hbar J_\pm$

$$\Leftrightarrow \boxed{J_z J_\pm = J_\pm (J_z \pm \hbar)}$$

we can write

$$J_z(J_\pm |a,b\rangle) = J_\pm (J_z \pm \hbar)|a,b\rangle = (b \pm \hbar)(J_\pm |a,b\rangle)$$

($J_z|a,b\rangle = b|a,b\rangle$)

$\Rightarrow J_\pm |a,b\rangle$ is also an eigenstate of J_z with eigenvalue $b \pm \hbar$.

Thus $J_\pm |a,b\rangle = C_\pm |a, b \pm \hbar\rangle$

($C_\pm$: scalar)