Thursday, October 22, 2026

Simultaneous eigenstates of $\vec{J}^2$ and J_z (and their eigenvalues)

$$\vec{J}^2 |a,b\rangle = a|a,b\rangle$$
$$J_z |a,b\rangle = b|a,b\rangle$$

Recall from last lecture: $J_\pm = J_x \pm i J_y$

$$[J_z, J_\pm] = \pm \hbar J_\pm$$
$$[\vec{J}^2, J_\pm] = 0$$
$$[J_+, J_-] = 2\hbar J_z$$

Since $[J_z, J_\pm] = \pm \hbar J_\pm \iff J_z J_\pm = J_\pm J_z \pm \hbar J_\pm$

$$\iff J_z J_\pm = J_\pm (J_z \pm \hbar \mathbb{1})$$

We can write

$$J_z (J_\pm |a,b\rangle) = J_\pm (J_z \pm \hbar \mathbb{1}) |a,b\rangle = (b \pm \hbar)(J_\pm |a,b\rangle)$$

(J_z acts with eigenvalue b; $\hbar\mathbb{1}$ with eigenvalue $\hbar$; together $b \pm \hbar$)

$\Rightarrow$ "$J_\pm |a,b\rangle$" is also an eigenstate of J_z with eigenvalue $b \pm \hbar$

$$J_\pm |a,b\rangle \propto |a, b\pm\hbar\rangle$$

Thus $J_\pm |a,b\rangle = C_\pm |a, b\pm\hbar\rangle$ ($C_\pm$: scalar)

Also, $\vec{J}^2\left(J_\pm |a,b\rangle\right) = J_\pm \vec{J}^2 |a,b\rangle = J_\pm \left(a|a,b\rangle\right)$
$= a\left(J_\pm |a,b\rangle\right)$

$\Rightarrow$ $J_\pm$ raises and lowers b, but does not affect a.

Next, $J^2 - J_z^2 = J_x^2 + J_y^2 = \frac{1}{2}\left(J_+ J_- + J_- J_+\right)$ (see calculation later)

$J_- = J_x - iJ_y = (J_x + iJ_y)^\dagger = J_+^\dagger$ (adjoint)
$= J_x^\dagger - iJ_y^\dagger$
$= J_x - iJ_y$

$= \frac{1}{2}\left(J_+ J_+^\dagger + J_+^\dagger J_+\right)$

We can use this equation:

$\langle a,b|\left(\vec{J}^2 - J_z^2\right)|a,b\rangle = \frac{1}{2}\langle a,b|J_+ J_+^\dagger|a,b\rangle + \frac{1}{2}\langle a,b|J_+^\dagger J_+|a,b\rangle$

$(a-b^2)\langle a,b|a,b\rangle$

$\langle\psi| \leftarrow |\psi\rangle$, $\langle\psi|\psi\rangle \geq 0$

$\langle\varphi| \leftarrow |\varphi\rangle$, $\langle\varphi|\varphi\rangle \geq 0$

≥ 0

$\Rightarrow$ $a - b^2 \geq 0$ $\Leftrightarrow$ $\boxed{a \geq b^2}$

↳ since b can be (increased / decreased) by ± 1 (repeatedly)

there must be a maximum b_{max} so that $J_+|a, b_{max}\rangle = 0$ rather than generating a state with $b^2 > a$

If $J_+|a, b_{max}\rangle = 0$, then $J_- J_+ (a, b_{max}\rangle = 0$

but $J_- J_+ = (J_x - iJ_y)(J_x + iJ_y) = J_x^2 + J_y^2 + i(J_x J_y - J_y J_x)$
$= J_x^2 + J_y^2 - \hbar J_z = \vec{J}^2 - J_z^2 - \hbar J_z$ ($i\hbar J_z$)

thus $J_- J_+ |a, b_{max}\rangle = 0$

$\Leftrightarrow \left(\vec{J}^2 - J_z^2 - \hbar J_z\right)|a, b_{max}\rangle = 0$

$\Leftrightarrow \left(a - b_{max}^2 - \hbar b_{max}\right)|a, b_{max}\rangle = 0$

not the null vector

$\Rightarrow a - b_{max}^2 - \hbar b_{max} = 0 \quad \Leftrightarrow \boxed{a = b_{max}(b_{max} + \hbar)}$

Similarly, there needs to be a b_{min}: $J_- |a, b_{min}\rangle = 0$

$\Rightarrow J_+ J_- |a, b_{min}\rangle = 0$

but $J_+ J_- = \vec{J}^2 - J_z^2 + \hbar J_z$

$\Rightarrow J_+ J_- |a, b_{min}\rangle = 0 \Leftrightarrow \left(J^2 - J_z^2 + \hbar J_z\right)|a, b_{min}\rangle = 0$

$\Leftrightarrow \left(a - b_{min}^2 + \hbar b_{min}\right)|a, b_{min}\rangle = 0$

not the null vector

$\Rightarrow a - b_{min}^2 + \hbar b_{min} = 0 \quad \Leftrightarrow \boxed{a = b_{min}(b_{min} - \hbar)}$

Thus, we have

$$b_{max}(b_{max} + \hbar) = a = b_{min}(b_{min} - \hbar)$$

which is solved by $\boxed{b_{max} = -b_{min}}$

Finally, if we act repeatedly on $|a, b_{min}\rangle$ with J_+, we get

$J_+ |a, b_{min}\rangle = J_+ |a, -b_{max}\rangle = (C_+)_{max} |a, -b_{max} + \hbar\rangle$

$J_+ |a, -b_{max} + \hbar\rangle = (C_+)_{max+\hbar} |-b_{max} + 2\hbar\rangle$

$\vdots$

$J_+ |a, \underbrace{-b_{max} + N\hbar}_{b_{max}}\rangle = 0$

$$b_{max} = -b_{max} + N\hbar \iff \boxed{b_{max} = \frac{N}{2}\hbar} \quad \text{where } N \in \mathbb{N}$$

$\Rightarrow$ thus possible values for b_{max} are $0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \dots$

We rename "$\frac{b_{max}}{\hbar}$" as "j", i.e. $j = \frac{b_{max}}{\hbar} = \frac{N}{2}$

In this case, we have $a = \hbar^2 j(j+1)$, and we set $b = m\hbar$, so that

$$\begin{cases} \vec{J}^2 |a,b\rangle \rightarrow \boxed{\vec{J}^2 |j,m\rangle = \hbar^2 j(j+1)|j,m\rangle} \\ J_z |a,b\rangle \rightarrow \boxed{J_z |j,m\rangle = \hbar m |j,m\rangle} \end{cases}$$

with $m = -j, -j+1, -j+2, \dots, j-2, j-1, j$

$|j,m\rangle$ are orthonormal, so $\boxed{\langle j',m'|j,m\rangle = \delta_{jj'}\delta_{mm'}}$

Next, recall that $\langle j,m|J_- J_+|j,m\rangle = (\langle j,m|J_+^\dagger)(J_+|j,m\rangle)$

$J_- J_+ = J^2 - J_z^2 - \hbar J_z$

$\langle j,m|J_+^\dagger = C_+^* \langle j,m+1|$; $J_+|j,m\rangle = C_+|j,m+1\rangle$

$C_+ = C_{+,jm}$

$$\iff \hbar^2 j(j+1) - m^2\hbar^2 - \hbar^2 m = |C_+|^2$$

$$\Rightarrow C_+ = \hbar\sqrt{j(j+1) - m^2 - m}$$

$$= \hbar\sqrt{j(j+1) - m(m+1)}$$

by convention we choose $C_+ \in \mathbb{R}$ (no complex phase)

thus $$\boxed{J_+|j,m\rangle = \hbar\sqrt{j(j+1) - m(m+1)}\ |j,m+1\rangle}$$

$$= \hbar\sqrt{(j-m)(j+m+1)}\ |j,m+1\rangle$$

similarly, $\boxed{J_- |j,m\rangle = \hbar\sqrt{j(j+1)-m(m-1)}\,|j,m-1\rangle}$

$= \hbar\sqrt{(j+m)(j-m+1)}\,|j,m-1\rangle$

Application #1 : $\boxed{j = 1/2}$, i.e. "spin-$\frac{1}{2}$"

$\Rightarrow m = 1/2$ and $-1/2$

basis : $\{|j,m\rangle\} = \{|1/2,+1/2\rangle, |1/2,-1/2\rangle\}$

$= \{|+\rangle, |-\rangle\}$

$= \{|\uparrow\rangle, |\downarrow\rangle\}$

$J^2 |1/2,\pm 1/2\rangle = \hbar^2 \tfrac{1}{2}(\tfrac{1}{2}+1)\,|1/2,\pm 1/2\rangle$

$= \hbar^2 \tfrac{3}{4}\,|1/2,\pm 1/2\rangle$

$J_z |1/2,\pm 1/2\rangle = \pm\frac{1}{2}\hbar\,|1/2,\pm 1/2\rangle \Rightarrow$ $\boxed{J_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$ (rows $|+\rangle$, $|-\rangle$; columns $|+\rangle$, $|-\rangle$)

$J_+ |1/2,+1/2\rangle = \hbar\sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1)-\tfrac{1}{2}(\tfrac{1}{2}+1)}\,|1/2,"3/2"\rangle = 0$ ($=0$)

$J_+ |1/2,-1/2\rangle = \hbar\sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1)+\tfrac{1}{2}(-\tfrac{1}{2}+1)}\,|\tfrac{1}{2},\tfrac{1}{2}\rangle = |1/2,+1/2\rangle$

$\sqrt{3/4+1/4} = \sqrt{1} = 1$