Tuesday, October 27, 2026 #1

**Angular Momentum** (reminder of formulas)

$$\vec{J}^2 |j,m\rangle = \hbar^2 j(j+1)\,|j,m\rangle \qquad J_z |j,m\rangle = \hbar m\,|j,m\rangle$$

$$J_+ |j,m\rangle = \hbar\sqrt{j(j+1)-m(m+1)}\,|j,m+1\rangle$$

$$J_- |j,m\rangle = \hbar\sqrt{j(j+1)-m(m-1)}\,|j,m-1\rangle$$

$m = -j, -j+1, \ldots, j-1, +j$

**Application #1:** $\boxed{j = 1/2}$ , i.e. "spin-$\frac{1}{2}$"

$\Rightarrow$ $m = 1/2$ and $-1/2$

basis: $\{|j,m\rangle\} = \{|1/2, +1/2\rangle, |1/2, -1/2\rangle\}$

$= \{|+\rangle, |-\rangle\}$

$= \{|\uparrow\rangle, |\downarrow\rangle\}$

$$J^2 |1/2, \pm 1/2\rangle = \hbar^2 \tfrac{1}{2}\left(\tfrac{1}{2}+1\right)|1/2, \pm 1/2\rangle = \hbar^2 \tfrac{3}{4}|1/2, \pm 1/2\rangle$$

$$J_z |1/2, \pm 1/2\rangle = \pm\tfrac{1}{2}\hbar\,|1/2, \pm 1/2\rangle \Rightarrow J_z = \frac{\hbar}{2}\begin{array}{c} \\ \langle +| \\ \langle -| \end{array}\begin{array}{c} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \begin{array}{cc} |+\rangle & |-\rangle \end{array} \end{array}$$

$$J_+ |1/2, +1/2\rangle = \hbar\sqrt{\tfrac{1}{2}\left(\tfrac{1}{2}+1\right)-\tfrac{1}{2}\left(\tfrac{1}{2}+1\right)}\,|1/2, \text{"}3/2\text{"}\rangle = 0$$

($=0$)

$$J_+ |1/2, -1/2\rangle = \hbar\sqrt{\tfrac{1}{2}\left(\tfrac{1}{2}+1\right)+\tfrac{1}{2}\left(-\tfrac{1}{2}+1\right)}\,|1/2, 1/2\rangle = \hbar\,|1/2, +1/2\rangle$$

$\sqrt{3/4 + 1/4} = \sqrt{1} = 1$

$$J_+ = \hbar \begin{matrix} \\ \langle +| \\ \langle -| \end{matrix} \begin{matrix} |+\rangle & |-\rangle \\ \end{matrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \longrightarrow \text{similarly, } J_- = \hbar \begin{matrix} \\ \langle +| \\ \langle -| \end{matrix} \begin{matrix} |+\rangle & |-\rangle \\ \end{matrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Also, $J_\pm = J_x \pm i J_y \Rightarrow \begin{cases} J_x = \frac{1}{2}(J_+ + J_-) \\ J_y = \frac{1}{2i}(J_+ - J_-) \end{cases}$

$$\Rightarrow J_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar}{2}\sigma_x \quad , \quad J_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix} = \frac{\hbar}{2}\sigma_y$$

$$\Rightarrow \boxed{J_i = \frac{\hbar}{2}\sigma_i = S_i} \text{ where } \sigma_i \text{ are the } \underline{\text{Pauli Spin matrices}}$$

## Rotation of a spin-$\frac{1}{2}$ state

Rotation by $\theta$ around the $x$-axis is thus

$$R_x(\theta) = e^{-i\frac{\theta}{\hbar}J_x} = e^{-i\frac{\theta}{\hbar}\sigma_x\frac{\hbar}{2}} = e^{-i\sigma_x\frac{\theta}{2}}$$

$$= \mathbb{1} + \frac{1}{1!}\sigma_x\left(-i\frac{\theta}{2}\right) + \frac{1}{2!}\sigma_x^2\left(-i\frac{\theta}{2}\right)^2 + \frac{1}{3!}\sigma_x^3\left(-i\frac{\theta}{2}\right)^3 + \cdots$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_x^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x^3 = \sigma_x, \quad \sigma_x^4 = \mathbb{1}, \ldots$$

$$\Rightarrow \sigma_x^{\text{n-even}} = \mathbb{1}, \quad \sigma_x^{\text{n-odd}} = \sigma_x$$

$$\Rightarrow R_x(\theta) = \mathbb{1} + \frac{1}{1!}\sigma_x\left(-i\frac{\theta}{2}\right) + \frac{1}{2!}\mathbb{1}\left(-i\frac{\theta}{2}\right)^2 + \frac{1}{3!}\sigma_x\left(-i\frac{\theta}{2}\right)^3 + \dots$$

$$= \mathbb{1}\underbrace{\left(1 - \frac{1}{2!}\left(\frac{\theta}{2}\right)^2 + \frac{1}{4!}\left(\frac{\theta}{2}\right)^4 - \frac{1}{6!}\left(\frac{\theta}{2}\right)^6 + \dots\right)}_{\cos(\theta/2)}$$

$$+ \sigma_x\left(-i\frac{\theta}{2} + \frac{i}{3!}\left(\frac{\theta}{2}\right)^3 - \frac{i}{5!}\left(\frac{\theta}{2}\right)^5 + \dots\right)$$

$$= \mathbb{1}\cos(\theta/2) - i\sigma_x\underbrace{\left(\frac{\theta}{2} - \frac{1}{3!}\left(\frac{\theta}{2}\right)^3 + \frac{1}{5!}\left(\frac{\theta}{2}\right)^5 - \dots\right)}_{\sin(\theta/2)}$$

$$= \mathbb{1}\cos(\theta/2) + \sigma_x(-i\sin\theta/2)$$

$$= \begin{bmatrix} \cos\theta/2 & -i\sin\theta/2 \\ -i\sin\theta/2 & \cos\theta/2 \end{bmatrix}$$

Thus $R_x(\theta) = e^{-i\frac{\theta}{\hbar}S_x} = e^{-i\sigma_x\theta/2} = \begin{bmatrix} \cos\theta/2 & -i\sin\theta/2 \\ -i\sin\theta/2 & \cos\theta/2 \end{bmatrix}$

We can use $R_x(\theta)$ to rotate a spin ket $(|\alpha\rangle = c_1|+\rangle + c_2|-\rangle)$ about the $x$-axis.

Note: $R_x(\theta = 2\pi) \neq \mathbb{1}$, but $R_x(\theta = 2\times 2\pi = 4\pi) = \mathbb{1}$

$$R_x(\theta = 2\pi) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad R_x(\theta = 4\pi) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

A spin-$\frac{1}{2}$ ket must be rotated by $4\pi$ to return to its original state!

# A few properties of Pauli spin matrices

1) Any $2\times 2$ Hermitian matrix $M$ can be written as a linear combination of the matrices $\{\mathbb{1}, \sigma_x, \sigma_y, \sigma_z\}$:

$$M = \alpha_0 \mathbb{1} + \alpha_x \sigma_x + \alpha_y \sigma_y + \alpha_z \sigma_z \quad \text{with } \alpha_0, \alpha_x, \alpha_y, \alpha_z \in \mathbb{R}$$
$$= \alpha_0 \mathbb{1} + \vec{\alpha}\cdot\vec{\sigma}$$

($\vec{\alpha}$: vector; $\vec{\sigma}$: vector of Pauli matrices)

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

2) $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \mathbb{1}$

3) Anti-commutator: $\{\sigma_i, \sigma_j\} = 2\,\delta_{ij}\,\mathbb{1}$

4) Commutator: $[\sigma_l, \sigma_m] = 2i\,\varepsilon_{lmn}\,\sigma_n$

5) $\sigma_i^\dagger = \sigma_i$ (Hermitian), $\mathrm{tr}(\sigma_i) = 0$, $\det(\sigma_i) = -1$

6) Important identity: $\vec{a}, \vec{b}$ = 3-vectors

$$\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$$

$$(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \vec{a}\cdot\vec{b}\,\mathbb{1} + i\,\vec{\sigma}\cdot(\vec{a}\times\vec{b})$$

proof (by Levi-Civita symbol)

$$(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = \left(\sum_j \sigma_j a_j\right)\left(\sum_k \sigma_k b_k\right) = \sum_{jk} \sigma_j \sigma_k\, a_j b_k$$

$$= \sum_{jk}\left(\frac{1}{2}\sigma_j\sigma_k + \frac{1}{2}\sigma_j\sigma_k + \frac{1}{2}\sigma_k\sigma_j - \frac{1}{2}\sigma_k\sigma_j\right) a_j b_k$$

$$= \sum_{jk}\left(\frac{1}{2}\underbrace{\{\sigma_j, \sigma_k\}}_{2\delta_{jk}\mathbb{1}} + \frac{1}{2}\underbrace{[\sigma_j, \sigma_k]}_{2i\varepsilon_{jkl}\sigma_l}\right) a_j b_k$$

$$= \sum_{jk} \left( \delta_{jk}\, a_j b_k \mathbb{1} + i \sum_l \varepsilon_{jkl}\, \sigma_l\, a_j b_k \right)$$

$$= \left( \sum_j a_j b_j \right) \mathbb{1} + i \sum_{jkl} \varepsilon_{jkl}\, a_j b_k\, \sigma_l$$

note: $(\vec{a}\times\vec{b})_l = \sum_{jk} \varepsilon_{jkl}\, a_j b_k$

$$= \vec{a}\cdot\vec{b}\,\mathbb{1} + \sum_l (\vec{a}\times\vec{b})_l\, \sigma_l$$

$$= \vec{a}\cdot\vec{b}\,\mathbb{1} + i(\vec{a}\times\vec{b})\cdot\vec{\sigma} = \vec{a}\cdot\vec{b}\,\mathbb{1} + i\,\vec{\sigma}\cdot(\vec{a}\times\vec{b})$$

Application: Consider a vector $\hat{n} = (n_1, n_2, n_3)$ with $\|\hat{n}\|^2 = 1$

$n_1, n_2, n_3 \in \mathbb{R}$

then $(\vec{\sigma}\cdot\hat{n})^2 = (\vec{\sigma}\cdot\hat{n})(\vec{\sigma}\cdot\hat{n}) =$

$$= \underbrace{\hat{n}\cdot\hat{n}}_{=1} + i\,\vec{\sigma}\cdot\underbrace{(\hat{n}\times\hat{n})}_{=0} = \mathbb{1}$$

$$(\vec{\sigma}\cdot\hat{n})^3 = (\vec{\sigma}\cdot\hat{n})(\vec{\sigma}\cdot\hat{n})^2 = (\vec{\sigma}\cdot\hat{n})\,\mathbb{1}$$

$$\Rightarrow (\vec{\sigma}\cdot\hat{n})^k = \begin{cases} \mathbb{1} & \text{for } k = \text{even} \\ \vec{\sigma}\cdot\hat{n} & \text{for } k = \text{odd} \end{cases}$$

Next, consider a rotation $\phi$ about an arbitrary direction/axis defined by $\hat{n}$:

$$R_{\hat{n}}(\phi) = e^{-i\frac{\vec{J}\cdot\hat{n}}{\hbar}\phi}$$

for a spinor (spin-$\frac{1}{2}$)

$$= e^{-i\frac{\vec{S}\cdot\hat{n}}{\hbar}\phi}$$

$$= e^{-i\,\vec{\sigma}\cdot\hat{n}\,\phi/2}$$

$$= 1 + \frac{(-i)}{1!}(\vec{\sigma}\cdot\hat{n})\left(\frac{\phi}{2}\right) + \frac{(-i)^2}{2!}(\vec{\sigma}\cdot\hat{n})^2\left(\frac{\phi}{2}\right)^2 + \frac{(-i)^3}{3!}(\vec{\sigma}\cdot\hat{n})^3\left(\frac{\phi}{2}\right)^3 + \frac{(-i)^4}{4!}(\vec{\sigma}\cdot\hat{n})^4\left(\frac{\phi}{2}\right)^4 + \cdots$$

$$= \mathbb{1}\left(1 - \frac{1}{2!}\left(\frac{\phi}{2}\right)^2 + \frac{1}{4!}\left(\frac{\phi}{2}\right)^4 - \dots\right) - i\left((\vec{\sigma}\cdot\hat{n})\left(\frac{\phi}{2}\right) - \frac{1}{3!}(\vec{\sigma}\cdot\hat{n})^3\left(\frac{\phi}{2}\right)^3 + \dots\right)$$

$\cos\left(\frac{\phi}{2}\right)$ $\qquad$ $(\vec{\sigma}\cdot\hat{n})\sin\left(\frac{\phi}{2}\right)$

$$= \cos\left(\frac{\phi}{2}\right)\mathbb{1} - i(\vec{\sigma}\cdot\hat{n})\sin\left(\frac{\phi}{2}\right)$$

$$\Rightarrow \boxed{R_{\hat{n}}(\phi) = \cos\left(\frac{\phi}{2}\right)\mathbb{1} - i\sin\left(\frac{\phi}{2}\right)(\vec{\sigma}\cdot\hat{n})}$$

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## Orbital Angular Momentum (see also Oct. 20 lecture)

$$\vec{L} = \vec{R}\times\vec{P} \quad , \quad [L_i, L_j] = i\hbar\,\epsilon_{ijk}L_k$$

Consider an infinitesimal rotation $d\phi$ about the z-axis

$$R_z(d\phi) = \mathbb{1} - i\frac{d\phi}{\hbar}L_z = 1 - i\frac{d\phi}{\hbar}(XP_y - YP_x)$$

we apply it to the 3D position ket $|\vec{r}\rangle = |x,y,z\rangle$

$$R_z(d\phi)|x,y,z\rangle = \left(1 - i\frac{d\phi}{\hbar}(XP_y - YP_x)\right)|x,y,z\rangle$$

$$= \left(1 - i\frac{x\,d\phi}{\hbar}P_y + i\frac{y\,d\phi}{\hbar}P_x\right)|x,y,z\rangle$$

$$= \left(1 - \frac{i}{\hbar}(-y\,d\phi, x\,d\phi, 0)\cdot(P_x, P_y, P_z)\right)|x,y,z\rangle$$

Translation operator for $d\vec{l} = (-y\,d\phi, x\,d\phi, 0)$

$$\begin{pmatrix} 1 & -d\phi & 0 \\ d\phi & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - y\,d\phi \\ y + x\,d\phi \\ z \end{pmatrix}$$

$$= |x - y\,d\phi, y + x\,d\phi, z\rangle$$

→ Clearly $L_z$ generates rotations around the z-axis