Tuesday, April 16, 2013

Quantum Mechanics was discovered through the following experiments/effects

- Black body radiation → photons
- Discrete spectrum of Hydrogen/elements → matter waves
- Photo-electric effect → photons
- Compton scattering ($\gamma - e^-$ scattering) → photons

Photon properties

$\vec{p} = \hbar \vec{k}$

$E = \hbar \omega$

What's the Hamiltonian for a photon?

What's the wavefunction for a photon?

PHOTONS: Quantization of the E-M field (PDQ)

(Sakurai & Napolitano, Loudon, Schiff)

In the absence of {charges, currents}, Maxwell's equations are

$$\begin{cases} \vec{\nabla}\cdot\vec{E} = 0 \\ \vec{\nabla}\times\vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \\ \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{B} = \dfrac{1}{c^2}\dfrac{\partial \vec{E}}{\partial t} \end{cases}$$

In the Coulomb gauge, $\vec{\nabla}\cdot\vec{A} = 0 \Rightarrow \nabla^2 V = 0$

$\Rightarrow$ set $V = 0$

In this case:
$$\begin{cases} \vec{E} = -\dfrac{1}{c}\dfrac{\partial \vec{A}}{\partial t} \\ \vec{B} = \vec{\nabla}\times\vec{A} \end{cases}$$
and $\vec{A}$ satisfies a wave equation: $\nabla^2 A - \dfrac{1}{c^2}\dfrac{\partial^2 A}{\partial t^2} = 0$

$\omega = ck$

$$\Rightarrow \text{Solutions: } \vec{A}_{k,s}(\vec{r},t) = \underbrace{A_{k,s}\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\, \hat{e}_{k,s} + A^*_{k,s}\, e^{-i(\vec{k}\cdot\vec{r}-\omega t)}\, \hat{e}_{k,s}}_{\text{real}}$$

to be specified

The general solution is: $\vec{A}(\vec{r},t) = \sum_{k,s} \vec{A}_{k,s}$ ← sum over discrete modes in box universe of volume V.

$\quad \vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \vec{\nabla}\times\vec{A}$

The energy and ~~Hamilt~~ "thus" Hamiltonian are given by

$$H_{\text{classical}} \equiv E = \frac{1}{2}\int_V d^3r \left\{ \varepsilon_0 \vec{E}^2 + \frac{1}{\mu_0}\vec{B}^2 \right\}$$

$H = \sum p_i \dot{q}_i - \mathcal{L}$

$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, $p_i = \dfrac{\partial \mathcal{L}}{\partial \dot{q}_i}$

where $q_i = A_i$

substitute for $\vec{E}$ & $\vec{B}$ ... after lots of algebra, we get

$$H_{classical} = \varepsilon_0 V \sum_{k,s} \omega_k^2 \left[A_{k,s} A^*_{k,s} + A^*_{k,s} A_{k,s} \right]$$

$$\int_V d^3 r\, e^{\pm i(k-k')\cdot r} = V \delta_{k,k'}$$

recall $H_{\text{quantum Harmonic oscillator}} = \frac{1}{2}\hbar\omega\left(a a^+ + a^+ a\right)$

we postulate that (quantization by analogy)

$$\boxed{H_{\text{photon field}} = \sum_{k,s} \frac{1}{2}\hbar\omega_{k,s}\left[a_{k,s} a^+_{k,s} + a^+_{k,s} a_{k,s} \right]}$$

bosonic creation and annihilation operators.

with $A_{k,s} = \sqrt{\dfrac{\hbar}{2\varepsilon_0 \omega_k V}}\, a_{k,s}$ and $A^*_{k,s} = \sqrt{\dfrac{\hbar}{2\varepsilon_0 \omega_k V}}\, a^+_{k,s}$

<u>note</u>: $H_{\text{photon field}} = \sum_{k,s} \cancel{\tfrac{1}{2}}\, \hbar\omega_{k,s}\left(a^+_{k,s} a + \frac{1}{2}\right)$

Fock state basis

the photon field wavefunction is the photon mode occupation number state

$$|\psi_{\text{photon field}}\rangle = |n_{k_1,s_+}, n_{k_1,s_-}, \ldots\ldots n_{k_N,s_+}, n_{k_N,s_-} \ldots\rangle$$

E-M field operators:

Heisenberg representation for the operators

$$\vec{E} = \sum_{k,s=1,2} \hat{\varepsilon}_{k,s}\left\{ i\omega_k A_{k,s}\, e^{i(k\cdot r - \omega_k t)} - i\omega_k A^*_{k,s}\, e^{-i(k\cdot r - \omega_k t)} \right\}$$

$$\vec{B} = \sum_{k,s=1,2} i k \times \hat{\varepsilon}_{k,s}\left\{ A_{k,s}\, e^{i(k\cdot r - \omega_k t)} - A^*_{k,s}\, e^{-i(k\cdot r - \omega_k t)} \right\}$$

Notable results for ~~B~~ photon Fock states

1. The average $\vec{E}$-field ($\vec{B}$-field) of a Fock state is zero
(ensemble average, not time average)

$$ {}_{\vec{k},s}\langle n | \vec{E} | n \rangle_{\vec{k},s} = 0 \quad , \text{but} \quad {}_{\vec{k},s}\langle n | \hat{N} | n \rangle_{\vec{k},s} = n $$

2. The variance ~~of~~ ΔE of the $\vec{E}$-field is always non-zero for a Fock state.

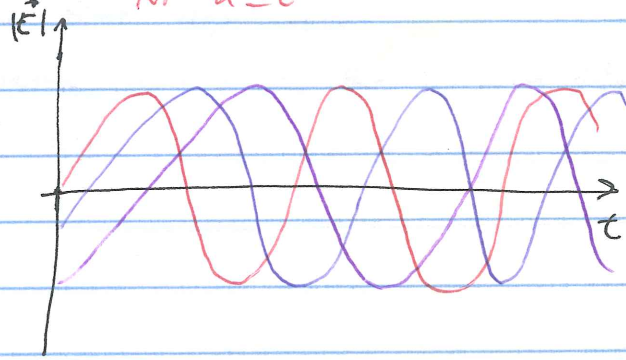
$$ \Delta E^2 = {}_{\vec{k},s}\langle n | \vec{E}^2 | n \rangle_{\vec{k},s} - \left(\cancel{{}_{\vec{k},s}\langle n | E | n \rangle_{\vec{k},s}}^{\,0} \right)^2 = \frac{\hbar \omega_k}{2 \varepsilon_0 V} (2n+1) $$

V ← quantization volume

$$ \boxed{\text{but } \Delta N^2 = \langle n | \hat{N}^2 | n \rangle - (\langle n | N | n \rangle)^2 = 0} $$

$$ \Rightarrow |\vec{E}| = \sqrt{\frac{\hbar \omega_k}{2 \varepsilon_0 V}} \sqrt{2n+1} \neq 0 \quad \text{even for } n = 0 $$

Physical picture of a ~~Fock~~ photon Fock state
→ superposition of classical $\vec{E}$-field plane waves but with random phases.



Coherent Photon States

Classical states of light (e.g. electromagnetic plane waves) are associated with coherent states $|\alpha\rangle_{\vec{k},s}$:

$$ \alpha \in \mathbb{C} ; \quad {}_{\vec{k},s}\langle \alpha | \alpha \rangle_{\vec{k},s} = 1 \quad ; \quad |\alpha\rangle_{\vec{k},s} = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle_{\vec{k},s} $$

A coherent state is an eigenstate of the annihilation operator:

$$ a_{\vec{k},s} |\alpha\rangle_{\vec{k},s} = \alpha |\alpha\rangle_{\vec{k},s} $$

$$ {}_{\vec{k},s}\langle \alpha | a^+_{\vec{k},s} = {}_{\vec{k},s}\langle \alpha | \alpha^* $$

⚠ coherent states are generally not orthogonal to each other ($\langle \alpha | \beta \rangle \neq 0$)

Properties: 1. $\langle n \rangle = {}_{\vec{k},s}\langle \alpha | N_{\vec{k},s} | \alpha \rangle_{\vec{k},s} = |\alpha|^2$

2_ $\Delta N^2 = {}_{\vec k,s}\langle\alpha|\, N_{\vec k,s}^2 \,|\alpha\rangle_{\vec k,s} - \left({}_{\vec k,s}\langle\alpha|\, N \,|\alpha\rangle_{\vec k,s}\right)^2 = \langle n\rangle$

$\Rightarrow \quad \Delta N = \sqrt{\langle n\rangle} \quad \Rightarrow$ | Poissonian statistics
The photon number fluctuates!

3_ ${}_{\vec k,s}\langle\alpha|\, \hat{E} \,|\alpha\rangle_{\vec k,s} = -\sqrt{\dfrac{\hbar\omega_k}{2\varepsilon_0 V}}\; 2|\alpha| \sin(\vec k\cdot\vec r - \omega_k t + \phi)\; \hat{\varepsilon}_{\vec k,s}$

$\alpha = |\alpha| e^{i\phi}$

This is a classical oscillating $\vec E$-field.

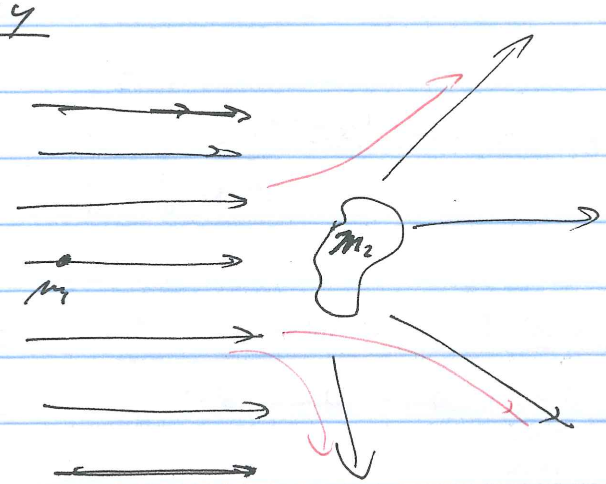
4_ $\Delta E^2 = \dfrac{\hbar\omega}{2\varepsilon_0 V} \quad \Rightarrow$
- independent of $|\alpha|$ and ϕ
- no time dependence
- same variance as the vacuum state!

Coherent state = classical state + vacuum noise

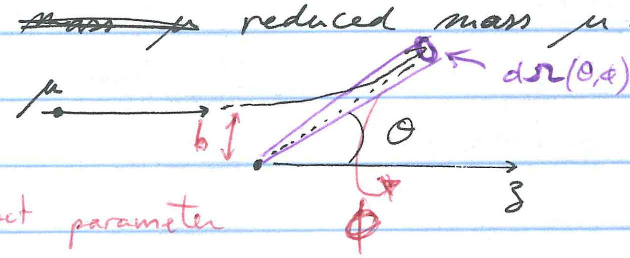
Stopped here

Quantum Scattering Theory

Classical scattering:



In the center-of-mass frame, we consider a particle of ~~mass μ~~ reduced mass μ: $\dfrac{1}{\mu} = \dfrac{1}{m_1} + \dfrac{1}{m_2}$



b = impact parameter