Tuesday, April 23, 2013

Quantum Scattering: Partial wave expansion (continued)

Recall: the asymptotic scattering eigenstates are of the form

$$\langle x | k, l, m=0 \rangle \underset{r \to +\infty}{\simeq} A \left[\frac{e^{-i(kr - l\pi/2)}}{r} - \frac{e^{i(kr - l\pi/2 + 2\delta_l)}}{r} \right] \times \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta)$$

Intuitively and experimentally, ~~the scat~~ we expect a scattering wavefunction of the form (in asymptotic limit)

$$\psi(\vec{r}) \underset{r \to +\infty}{=} \underbrace{e^{ikz}}_{\text{incident plane wave}} + f(\theta; k) \underbrace{\frac{e^{ikr}}{r}}_{\text{outgoing spherical wave}}$$

decompose on scattering eigenstates

$$= \frac{i}{2k} \sum_{l=0}^{\infty} i^l (2l+1) \left[\frac{e^{-i(kr - l\pi/2)}}{r} - \underbrace{\left[1 + 2ik\, f_l(k)\right]}_{e^{i2\delta_l}} \frac{e^{i(kr - l\pi/2)}}{r} \right] P_l(\cos\theta)$$

Amplitude must have norm = 1 to ensure conservation of probality (particle #) per angular momentum channel

with $f(\theta; k) = \frac{1}{2ik} \sum_{l=0}^{\infty} (2l+1)\, 2ik\, f_l(k)\, P_l(\cos\theta) = \sum_{l=0}^{\infty} (2l+1) f_l(k) P_l(\cos\theta)$

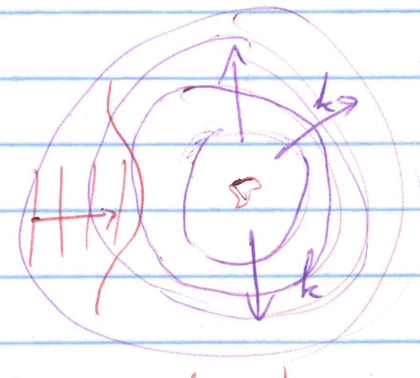
thus $f(\theta;k) = \sum_{\ell=0}^{\infty} \frac{(2\ell+1)}{2ik} \underbrace{\left(e^{i2\delta_\ell}-1\right)}_{2ik\, f_\ell(k)} P_\ell(\cos\theta)$

$$\Rightarrow \quad f_\ell(k) = \frac{e^{i2\delta_\ell}-1}{2ik}$$

Differential cross-section

$$\psi(\vec r) \underset{r\to+\infty}{=} A(x,y)\, e^{ikz} + f(\theta;k)\frac{e^{ikr}}{r}$$

plane wave with
an envelope so that
~~beyond~~ for x,y sufficiently far from
z-axis, $A(x,y) \longrightarrow 0$
(implies a spread of k, but this will be [illegible])



outside ~~$\vec{k}(\vec r)$~~ of $A(x,y)$ incident plane wave envelope, we have

allows us to ignore interference between incident and outgoing waves

$$\psi(\vec r) = \psi_{scattered}(\vec r) = f(\theta;k)\frac{e^{ikr}}{r}$$

$$\Rightarrow \quad P(\theta;k) = \left|\psi_{scattered}(\vec r)\right|^2 = \frac{1}{r^2}\left|f(\theta;k)\right|^2$$

must have units of area

↳ Probability to get a scattered particle in solid angle $d\Omega$:

$$P(\theta;k)\, r^2 d\Omega = \left|f(\theta;k)\right|^2 d\Omega$$

$\sin\theta\, d\theta\, d\phi$

$$\Rightarrow \quad \boxed{\text{differential cross-section} = \frac{d\sigma}{d\Omega} = \left|f(\theta;k)\right|^2}$$

alternate derivation $\vec{j}$ = probability current $= \frac{1}{\mu}\mathrm{Re}\left[\psi^*_{scattered}\frac{\hbar}{i}\nabla\psi_{scattered}\right]$

total cross-section

$$\sigma_{total} = \int_\Omega \frac{d\sigma}{d\Omega}\, d\Omega = \int_0^{2\pi}\int_0^{\pi} |f(\theta;k)|^2\, \underset{\uparrow\ \sin\theta\, d\theta\, d\phi}{d\Omega}$$

$$= 2\pi \int_0^{\pi} |f(\theta;k)|^2 \sin\theta\, d\theta$$

$$f(\theta;k) = \sum_{\ell=0}^{\infty} (2\ell+1)\, \frac{e^{i2\delta_\ell}-1}{2ik}\, P_\ell(\cos\theta)$$

$$= \sum_{\ell=0}^{\infty} (2\ell+1) e^{i\delta_\ell}\, \frac{1}{k}\, \underbrace{\frac{e^{i\delta_\ell}-e^{-i\delta_\ell}}{2i}}_{\sin\delta_\ell}\, P_\ell(\cos\theta)$$

$$= \sum_{\ell=0}^{\infty} (2\ell+1)\, e^{i\delta_\ell}\, \frac{\sin\delta_\ell}{k}\, P_\ell(\cos\theta)$$

Thus

$$\sigma_{total} = 2\pi \int_0^{\pi} \sum_{\ell=0}^{\infty}\sum_{\ell'=0}^{\infty} (2\ell+1)\cancel{(2\ell'+1)}\, e^{i\delta_\ell} e^{-i\delta_{\ell'}}\, \frac{\sin\delta_\ell}{k}\, \frac{\sin\delta_{\ell'}}{k}\, P_\ell(\cos\theta) P_{\ell'}(\cos\theta) \sin\theta\, d\theta$$

integrate

$$= \frac{2}{\cancel{2\ell+1}}\, \delta_{\ell\ell'}$$

$$\Rightarrow \quad \boxed{\sigma_{total} = 4\pi \sum_{\ell=0}^{\infty} (2\ell+1)\, \frac{\sin^2\delta_\ell}{k^2}}$$

$\delta_\ell(k)$

Determining δ_ℓ → read Sakurai & Napolitano p414–417
Sakurai (red book) p405–408

Low Energy Scattering

$V(r)$, the scattering potential, has a finite range (i.e. $V(r)=0$ for $r > R_0$). ~~For~~ ~~at~~ Semi-classically, ~~for~~ for a given energy, $E = \frac{(\hbar k)^2}{2\mu}$, the maximum angular momentum to participate in collision is $(\hbar k) R_0 \sim \hbar \sqrt{\ell_{max}(\ell_{max}+1)}$

$p \times r = L$

If we choose $\ell_{max} = 0$, then for $k \ll \frac{1}{R_0}$, we ~~sh~~ expect $\delta_{\ell>0} = 0$ and $\delta_{\ell=0} \neq 0$ to completely determine the low energy scattering properties.

ex: - ultra cold atom collisions
- low energy $e^- +$ atom collisions
- low energy $n+p$ scattering.

In the asymptotic limit, the schrodinger equation becomes

$$-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2}u_{\ell=0}(r) \simeq \frac{\hbar^2 k^2}{2\mu} \xrightarrow[\ell=0]{k\to 0} \frac{d^2}{dr^2}u \sim 0$$

$\psi(\vec{r}) = \frac{u_{E,\ell}(r)}{r} Y_\ell^m(\theta,\phi)$

$$\Rightarrow \quad u \underset{k\to 0}{\simeq} cst\,(r - \overset{cst}{a_s})$$

$=$ straight line

However, we also know that in the asymptotic limit

$$u_{\substack{\ell=0\\k\to 0}} \sim \sin\left(kr - \cancel{\ell \tfrac{\pi}{2}} + \delta_{\ell=0}\right)$$

$$\sim \sin\left[k\left(r + \frac{\delta_{\ell=0}}{k}\right)\right]$$

$$\sim k\left(r + \frac{\delta_{\ell=0}}{k}\right) \sim cst\,(r - a_s)$$

we identify $a_s \simeq \lim_{k\to 0} -\frac{\delta_{\ell=0}}{k}$

more formally $r - a_s = \lim_{k\to 0} \frac{u}{\frac{du}{dr}} = \lim_{k\to 0} \frac{\sin\left[k\left(r + \frac{\delta_{\ell=0}}{k}\right)\right]}{k\cos\left[k\left(r + \frac{\delta_{\ell=0}}{k}\right)\right]}$

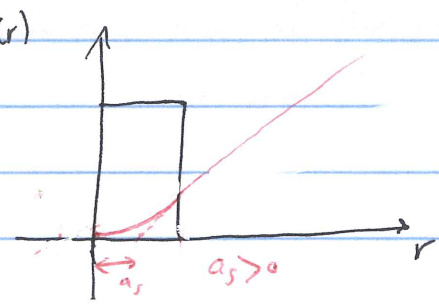
$$r - a_s = \lim_{k\to 0} \frac{1}{k} \tan\left[k\left(r + \frac{\delta_{\ell=0}}{k}\right)\right]$$

if we pick $r=0$, then $\boxed{a_s = -\lim_{k\to 0} \frac{\tan(\delta_{\ell=0})}{k}}$

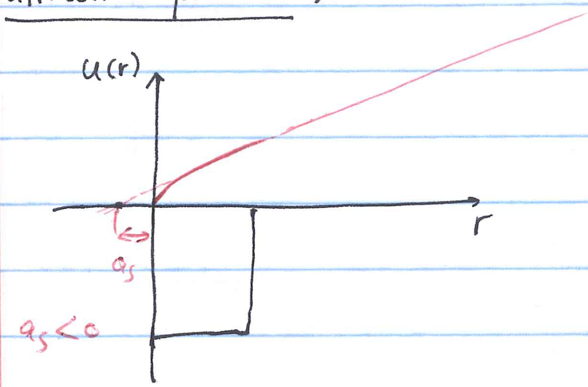
a_s is called the scattering length and completely characterizes low energy scattering : $\delta_{\ell=0} \simeq -a_s k$

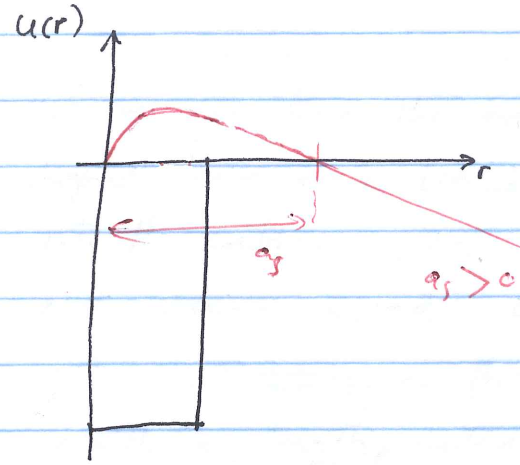
physically, a_s is the low-k intercept of wavefunction $(u(r))$ with the r-axis

repulsive potential



attractive potentials





Note: For attractive potentials, a_s is very sensitive to the exact form of $V(r)$, and is thus too difficult to calculate *ab initio* (exception hydrogen).

In the s-wave limit, $\delta_{l=0} \doteq -a_s k$ and $\delta_{l\neq 0} = 0$, so

$$\sigma_{total} = 4\pi \sum_{l=0}^{\infty} (2\cancel{l}^{\,0}+1) \underbrace{\frac{\sin^2(-a_s k)}{k^2}}_{(\text{for } k\to 0)\ a_s^2} \Rightarrow \underset{k\to 0}{\boxed{\sigma_{total} = 4\pi a_s^2}}$$

Ramsauer-Townsend Effect:

In the s-wave limit, if k is increased a little, then ~~$\delta_{l=0}(k$~~ for some $k \neq 0$ but ~~#~~ still small, we get $\delta_{l=0} = \pm\pi$ $= \pm n\pi$

so that $\sigma_{total} = 4\pi \dfrac{\sin^2(\delta_{l=0})}{k^2} = \dfrac{4\pi}{k^2} \cancel{\sin^2(n\pi)}^{\,0} = 0$.

In this case there, ~~is no sea~~ at this particular k (i.e. energy) then there is no scattering → the particles are "transparent" to each other.