Thursday, April 25, 2013

Low Energy scattering (continued)

At low energy, as the collision momentum (in C.M.) $k \to 0$, only the $l=0$ (s-wave) character of the particles participates in the collision ~~and~~ The collision is well described by a single parameter, the scattering length a_s :

$$a_s(V) = \lim_{k\to 0} - \frac{\tan \delta_{l=0}(k)}{k} \Rightarrow \delta_{l=0} \underset{k\to 0}{\simeq} -a_s k$$

inter-particle potential

Thus the total cross-section for $k \to 0$, reduces to

$$\sigma_{total} = 4\pi \sum_{l=0}^{\infty} (2\cancel{l}^{\,0}+1) \frac{\sin^2(\delta_l)^{\,a_s^2}}{\cancel{k^2}} = 4\pi a_s^2$$

Also, $f(\theta;k) = \sum_{l=0}^{\infty} (2l+1) f_l(k) P_l(\cos\theta)$

for $k \to 0$ $\quad f(\theta;k) = \underbrace{f_0(k)} \cancel{P_{l=0}}^{\,1}(\cos\theta)$

$$\frac{e^{i2\delta_{l=0}}-1}{2ik} = \frac{e^{i\delta_{l=0}}}{k} \frac{e^{i\delta_{l=0}} - e^{-i\delta_{l=0}}}{2i}$$

$$= e^{-ia_s k} \frac{\sin(-a_s k)}{k}$$

$$= \lim_{k\to 0} e^{-ia_s k} \frac{\sin(-a_s k)}{k} = -a_s$$

$$\Rightarrow \boxed{f(\theta, k\to 0) = -a_s}$$

thus in the $k \to 0$ (s-wave limit) the scattering wave function reduces to

$$\psi(\vec{r}) \underset{k\to 0}{=} \cancel{\ } e^{ikz} + f(\theta, k)\frac{e^{ikr}}{r} \simeq 1 - \frac{a_s}{r}$$

Effective Range of s-wave scattering

In the s-wave limit, $f(\theta, k) = \cancel{(2l+1)} f_{l=0}(k)$

$$= \frac{e^{i\delta_{l=0}} \sin(\delta_{l=0})}{k}$$

$$= \left[\cos(\delta_{l=0}) + i\sin(\delta_{l=0})\right]\frac{\sin(\delta_{l=0})}{k}$$

$$\times\frac{(\cos\delta - i\sin\delta)}{(\cos\delta - i\sin\delta)}$$

$$= \frac{1}{k}\,\frac{\sin\delta_{l=0}/\sin\delta}{\cos\delta_{l=0} - i\sin\delta_l/\sin\delta} = \frac{1}{k\,\mathrm{cotan}\,\delta_{l=0} - ik}$$

We can rewrite the scattering length condition

$$a_s = \lim_{k\to 0} -\frac{\tan(\delta_{l=0})}{k} \quad \text{as} \quad -\frac{1}{a_s} = \lim_{k\to 0} k\,\mathrm{cotan}(\delta_{l=0})$$

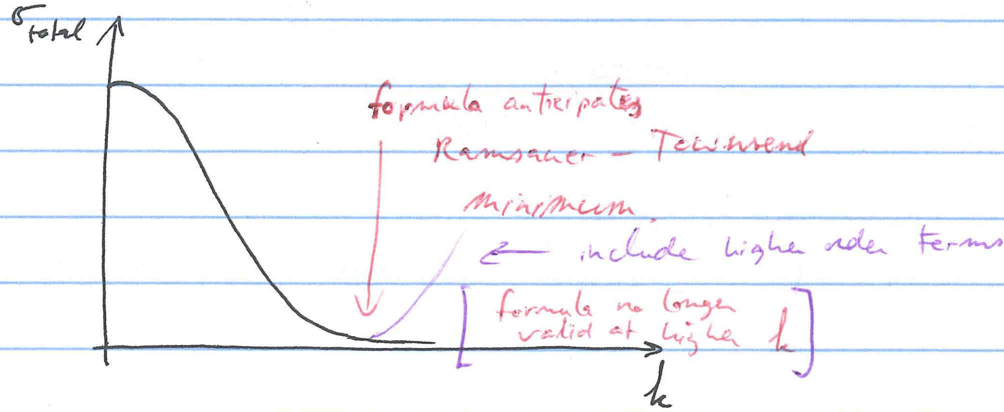
In fact, $k\,\mathrm{cotan}(\delta_{l=0}) = -\frac{1}{a_s} + \frac{1}{2} r_0 k^2 + \ldots$ (Taylor expans.)

r_0 = effective range (s-wave) of scattering potential

thus $\sigma_{total} = \int_\Omega |f(\theta, k)|^2 d\Omega = 4\pi \left|f_{l=0}(k)\right|^2$

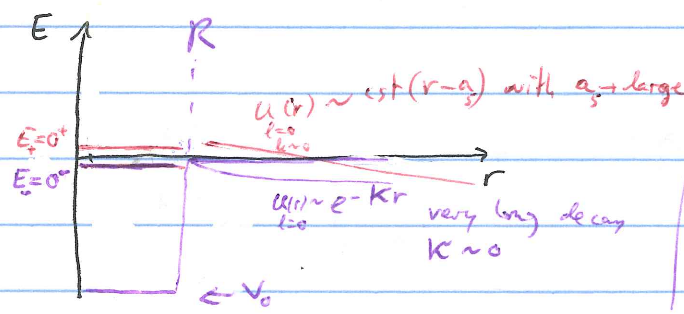
$$= \frac{4\pi}{\left|\left(-\frac{1}{a_s} + \frac{1}{2} r_0 k^2\right) - ik\right|^2} = \frac{4\pi}{\left(-\frac{1}{a_s} + \frac{1}{2} r_0 k^2\right)^2 + k^2}$$

so $\boxed{\sigma_{Total} = \dfrac{4\pi a_s^2}{\left(1 - \frac{1}{2} a_s r_0 k^2\right)^2 + a_s^2 k^2}}$ is a more accurate description of low energy s-wave scattering



Scattering Length and Bound states (see Sakurai 7.7, 6.6)

The scattering length is closely related to the binding energy of the highest bound state



for $r < R$, the schrodinger equation is

$$\left(\frac{-\hbar^2}{2\mu}\frac{d^2}{dr^2} + V_0\right) u_{\substack{l=0\\k}}(r) = E_\pm u_{\substack{l=0\\k}}(r)$$

k^2 $\qquad$ ~ 0

$\Rightarrow \dfrac{\hbar^2 k^2}{2\mu} \simeq V_0$ for the barely bound state and the barely free state

for $r > R$

$E_+ = 0^+ \rightarrow$ barely free state $\quad u_{\substack{l=0\\k\sim 0}}(r) \sim cst(r - a)$

$E_- = 0^- \rightarrow$ barely bound state $\quad u_{\substack{l=0\\K\sim 0}}(r) \sim e^{-Kr}$ with $K \sim 0$ very long decay boundary conditions: $\frac{d}{dr}u(r=R^-) = \frac{d}{dr}u(r=R^+)$ continuous derivative

$u(r=R^-) = u(r=R^+)$ continuous function

$$\Rightarrow \left.\frac{\frac{d}{dr}u(r=R^-)}{u(r=R^-)}\right|_{E=0_\pm} = \left.\frac{\frac{d}{dr}u(r=R^+)}{u(r=R^+)}\right|_{E=0_\pm}$$

However we expect $\left.\frac{\frac{d}{dr}u(r=R^-)}{u(r=R^-)}\right|_{E=0_+} \simeq \left.\frac{\frac{d}{dr}u(r=R^-)}{u(r=R^-)}\right|_{E=0_-}$

since $\underset{E=0_-}{u(r<R)} \simeq \underset{E=0_+}{u(r<R)}$ i.e, inside wave functions are roughly the same for nearly bound/free

thus $\left.\frac{\frac{d}{dr}u(r=R^+)}{u(r=R^+)}\right|_{E=0_+} \simeq \left.\frac{\frac{d}{dr}u(r=R^+)}{u(r=R^+)}\right|_{E=0_-}$

$$\Leftrightarrow \left.\frac{\cancel{cst}}{\cancel{cst}\,(r-a_s)}\right|_{r=R^+} = \left.\frac{-\kappa\,\cancel{e^{-\kappa r}}}{\cancel{e^{-\kappa r}}}\right|_{r=R^+}$$

$\Rightarrow$ if $R^+ \ll a$ then $\boxed{\kappa \simeq \frac{1}{a_s}}$

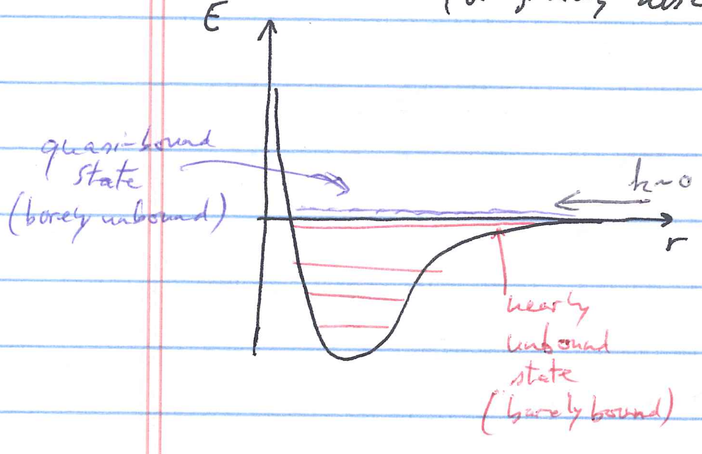
However $E_{\text{binding energy}} \sim -0_- \simeq \frac{\hbar^2\kappa^2}{2\mu}$

$$\Rightarrow \boxed{E_{\text{binding energy}} = \frac{\hbar^2}{2\mu}\frac{1}{a_s^2}}$$

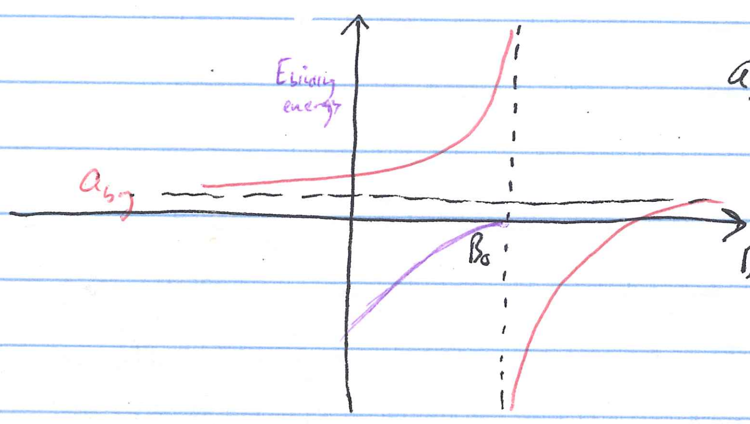
choose "+"

Note: $a_s = +\sqrt{\frac{\hbar^2}{2\mu}\frac{1}{E_{\text{binding energy}}}}$ so $\lim_{E_{\text{binding energy}}\to 0} a_s \to +\infty$

Feshbach Resonances: scattering from quasi-bound states.
(originally discovered in nuclear scattering)



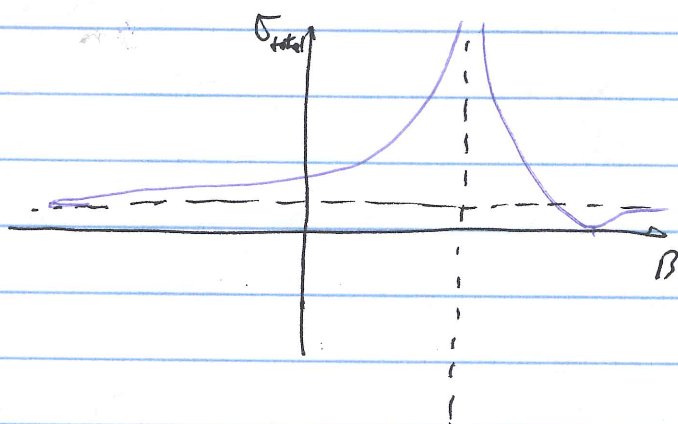
If we can tune the ~~bound~~ highest bound state from bound into the continuum (e.g. using ~~a~~ the Zeeman effect) then we get a resonance in the scattering length, called a Feshbach resonance



$$a_s = a_{bg}\left(1 - \frac{\Delta}{B - B_0}\right)$$

Δ: magnetic width of resonance

you can derive this from spherical square well model [replace B with V_0] [$B_0 \to V_{0,\text{critical}}$ where new bound state appears]



Applications: - Control collision ~~[illegible]~~ properties with magnetic field
- Ultracold molecule production

Identical Particle collisions

identical bosons: spatial wavefunction must be symmetric under particle exchange
(identical fermions) (anti-symmetric)

$$\psi(\vec{r}) = e^{ikz} \pm f(\theta)\,\frac{e^{ikr}}{r}$$

$\downarrow$ symmetrize

$$\psi_{boson}(\vec{r}) = e^{ikz} \pm e^{-ikz} + \left[f(\theta) \pm f(\pi-\theta)\right]\frac{e^{ikr}}{r}$$

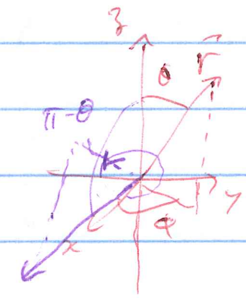
particle exchange: $z \to -z$

$\left(\begin{array}{l} x \to -x \\ y \to -y \end{array}\right)$

(or $k \to -k$)

particle exchange: $\vec{r} \to -\vec{r}$

$\longrightarrow \begin{cases} r \to r \\ \theta \to \pi - \theta \\ \phi \to \phi + \pi \end{cases}$



note! $\sigma = \int_{\text{final states}} \frac{d\sigma}{d\Omega}\, d\Omega = \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta\, \frac{d\sigma}{d\Omega} \sin\theta$ ← avoids double counting

$$\Rightarrow \frac{d\sigma}{d\Omega} = |f(\theta)|^2 \rightarrow \frac{d\sigma}{d\Omega} = \left|f(\theta) \pm f(\pi-\theta)\right|^2$$

In the s-wave limit ($k \to 0$), $\begin{cases} f(\theta) \simeq -a_s \\ f(\pi-\theta) \simeq -a_s \end{cases}$

$$\Rightarrow \sigma_{total} \underset{k\to 0}{=} \begin{cases} 2\pi\left|-a_s + (-a_s)\right|^2 = 8\pi a_s^2 \\ 2\pi\left|-a_s - (-a_s)\right|^2 = 0 \end{cases}$$

→ identical bosons interact and collide with double the strength

← identical fermions don't collide!

More generally, identical Bosons collide with $l = 0, 2, 4, 6, \ldots$

identical fermions collide with $l = 1, 3, 5, \ldots$