Thursday, January 31, 2013 #1

Hydrogen with lowest order relativistic correction to the kinetic energy

$$H = H_0 - \frac{P^4}{8 m_0^3 c^2} \qquad \left(H_0 = \frac{P^2}{2m_0} - \frac{e^2}{R}\right)$$

$\Rightarrow$ there is no zeroth order correction to eigenvector states

the 1st order correction to the energy is

$$E_{nlm} = E_n + \langle n\, l\, m_l | W | n\, l\, m_l \rangle$$

$$= E_n + \langle n\, l\, m_l | \frac{-P^4}{8 m_0^3 c^2} | n\, l\, m_l \rangle$$

(4th order derivatives)

trick: $$\frac{P^4}{8 m_0^3 c^2} = \frac{1}{2 m_0 c^2}\left(\frac{P^2}{2 m_0}\right)^2 = \frac{1}{2 m_0 c^2}\left(H_0 + \frac{e^2}{R}\right)^2$$

$$\langle n,l,m_l | \frac{-P^4}{8 m_0^3 c^2} | n,l,m_l \rangle = \frac{-1}{2 m_0 c^2} \langle n,l,m_l | \left(H_0 + \frac{e^2}{R}\right)^2 | n,l,m_l \rangle$$

$$= \frac{-1}{2 m_0 c^2}\Big\{ \underbrace{\langle n,l,m_l | H_0^2 | n,l,m_l \rangle}_{E_n^2} + \underbrace{\langle n,l,m_l | H_0}_{E_n \langle n,l,m_l|} \frac{e^2}{R} | n,l,m_l \rangle$$

$$+ \langle n,l,m_l | \frac{e^2}{R} \underbrace{H_0 | n,l,m_l \rangle}_{E_n |n,l,m_l\rangle} + \langle n,l,m_l | \frac{e^4}{R^2} | n,l,m_l \rangle \Big\}$$

$E_n = -\dfrac{e^2}{2 n^2 a_0}$ where $e^2 = \dfrac{q_e^2}{4\pi\epsilon_0}$

$$= -\frac{1}{2 m_0 c^2}\Big\{ E_n^2 + 2 E_n \underbrace{\langle n,l,m_l | \frac{e^2}{R} | n,l,m_l \rangle}_{\frac{e^2}{n^2 a_0} = -2E_n} + \underbrace{\langle n,l,m_l | \frac{e^4}{R^2} | n,l,m_l \rangle}_{\frac{e^4}{n^3 a_0^2 (l+\frac{1}{2})} = \frac{4n}{l+\frac{1}{2}} E_n^2}$$

see Sakurai Appendix on H atom

thus $\langle n, l, m_l | \frac{-P^4}{8 m_o^3 c^2} | n, l, m_l \rangle$

$$= -\frac{1}{2}\frac{1}{m_o c^2}\left\{ -3E_n^2 + \frac{4n}{l+\frac{1}{2}} E_n^2 \right\}$$

$$= -\frac{E_n^2}{2 m_o c^2}\left\{ -3 + \frac{4n}{l+\frac{1}{2}} \right\}$$

$\rightarrow$ fractional correction $\frac{E_n}{m_e c^2} \sim \frac{10\,eV}{500\,keV} \sim 0.02 \times 10^{-3} \sim 10^{-5}$

note: l - degeneracy lifted

m_l - degeneracy remains

Scale of perturbation:

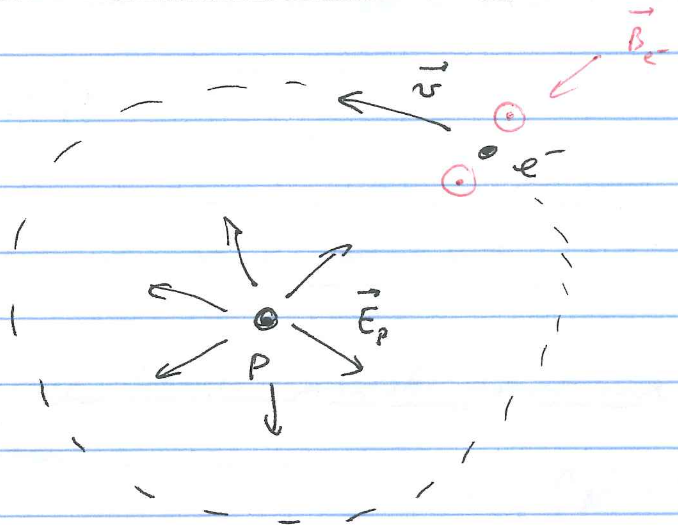
$$\frac{\frac{P^4}{8 m_e^3 c^2}}{\frac{P^2}{2 m_e}} = \frac{1}{2}\,\frac{P^2}{2 m_e} \Big/ m_e c^2$$

$$\sim \frac{1}{2}\,\frac{10\,eV}{500\,keV}$$

$$\sim \frac{1}{100 \times 10^{+3}} = 10^{-5}$$

Spin - Orbit Interaction and Fine structure

Electric - field in Lab/proton reference frame seen by e^-:



potential

$$\vec{E} = -\nabla V = -\frac{\partial V}{\partial r}\hat{r}$$

$$= -\frac{|q_e|}{4\pi\epsilon_0}\left(-\frac{1}{r^2}\right)\hat{r} = \frac{|q_e|}{4\pi\epsilon_0 r^2}\hat{r}$$

$V = \frac{q_e}{4\pi\epsilon_0 r}$

(not potential energy)

Magnetic - field in lab/proton reference frame seen by e^-

$$\vec{B} = 0$$

Electric - field of p in e^- reference frame

$$\vec{E}_{\parallel,e^-} = \vec{E}_{\parallel} = \vec{E}\cdot\frac{\vec{v}}{|\vec{v}|}, \quad \vec{E}_{\perp,e^-} = \gamma\vec{E}_{\perp} = \text{~~[struck out]~~}$$

where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

Magnetic - field in e^- reference frame

$$\vec{B}_{e^-} = -\frac{1}{c^2}\vec{v}\times\vec{E}_{e^-} = -\frac{1}{c^2}\vec{v}\times\gamma\vec{E}_{\perp}$$

for $\frac{v}{c} \ll 1$

$\gamma \cong 1 + \frac{1}{2}\frac{v^2}{c^2}$

Important: Classical Electrodynamics is relativistically invariant.

↳ it gives the same answer/trajectory for <u>charges</u> regardless of reference frame.

This is not true for an object with a magnetic moment i.e. a spin.

Magnetic moment review

The force on a magnetic dipole $\vec{M}$ is given by $\vec{F} = \nabla(\vec{M}\cdot\vec{B})$

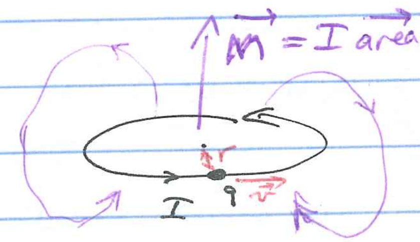
The potential energy of a magnetic dipole $\vec{M}$ in a magnetic field $\vec{B}$ is thus $U = -\vec{M}\cdot\vec{B}$.

$$\Rightarrow \quad \boxed{H_{int} = -\vec{M}\cdot\vec{B}}$$

Q: What's the magnetic moment of an e^- or proton?

alternate Q: what's the relationship between magnetic moment and angular momentum?

Classical model of a magnetic moment



$$I = \frac{q}{\text{period}} \qquad \text{with period} = \frac{2\pi r}{v}$$

$$= \frac{qv}{2\pi r}$$

$$\vec{\text{Area}} = \pi r^2 \hat{n}$$

$$\text{thus } \vec{M} = \frac{qv}{2\pi r}\pi r^2 \hat{n} = \frac{q}{2m}\underbrace{mvr\,\hat{n}}_{\text{angular momentum } \vec{l}} = \frac{q}{2m}\vec{l}$$

More generally one might expect this treatment to be accurate up to an overall factor (fudge factor), when considering the intrinsic spin of a particle

$$\boxed{\vec{M} = \frac{q}{2m}\vec{L} \qquad \text{but} \qquad \vec{M} = g\frac{q}{2m}\vec{S}}$$

g = g-factor

$g_e = 2.002319304$ (from QED & Experiment)

$g_{e^-} \simeq 2.0 \quad , \quad g_p \simeq 5.6 \quad , \quad g_n \simeq -3.8$

$g_\mu \simeq 2.0$

<u>note</u>: $\dfrac{|\vec{M}_e|}{|\vec{M}_{p,n}|} \sim \dfrac{m_{p,n}}{m_e} \sim 1800$

$\Rightarrow$ magnetic moment of an atom is completely dominated by $\vec{M}_e$.

Also, one frequently writes $\vec{M}_e = g_e \dfrac{q_e \hbar}{2m}\left(\dfrac{\vec{S}}{\hbar}\right) = g_e \mu_B \left(\dfrac{\vec{S}}{\hbar}\right)$

also $\vec{M}_e = 2.0 \dfrac{q_e \vec{S}}{2m_e} = \dfrac{q_e \vec{S}}{m_e}$	with $\mu_B = \dfrac{q_e \hbar}{2m} = -9.274 \times 10^{-24}\ J/T$ $= 1.4\ MHz/G$ $=$ Bohr magneton

<u>Back to Spin-Orbit coupling in H</u>

The B-field $\vec{B}_{e^-} = -\dfrac{1}{c^2}\vec{v} \times \vec{E}_{e^-}$ in the e^-'s instantaneous reference frame couples to the e^-'s magnetic moment.

So we expect $H_{int} = H_{\text{spin orbit}} = -\vec{M}_e \cdot \vec{B}_{e^-}$

$$= -\frac{q_e \vec{S}}{m_e} \cdot \left(-\frac{1}{c^2}\right) \vec{v} \times \left(\frac{|q_e|}{4\pi\epsilon_0} \gamma \frac{\hat{r}}{r^2}\right)$$

$\dfrac{q_e^2}{4\pi\epsilon_0} = e^2$

$$= -\frac{e^2 \gamma}{m_e^2 c^2} \frac{1}{r^3} \vec{S} \cdot (\vec{p} \times \vec{r})$$

($\vec{p} \times \vec{r} = -\vec{\ell}$)

$$= \frac{e^2 \gamma}{m_e^2 c^2} \frac{1}{r^3} \vec{S} \cdot \vec{\ell}$$

If we quantize the observables, then

$$H_{\substack{spin\\orbit}} = \frac{e^2}{m_e^2 c^2}\,\gamma\,\frac{1}{R^3}\,\vec{L}\cdot\vec{S} \qquad \text{note } [\vec{L},\vec{S}]=0$$

to lowest order in $\frac{v}{c}$, ~~the~~ $\gamma \to 1$

$$\simeq \frac{e^2}{m_e^2 c^2}\,\frac{1}{R^3}\,\vec{L}\cdot\vec{S}$$

when compared with experiment, one finds that this expression is a factor of 2 too large.

$$\boxed{H_{\substack{spin\\orbit}} = \frac{1}{2}\,\frac{e^2}{m_e^2 c^2}\,\frac{1}{R^3}\,\vec{L}\cdot\vec{S}}$$

$$= \underbrace{\frac{e^2}{m_e^2 c^2}\,\frac{1}{R^3}\,\vec{L}\cdot\vec{S}}_{H_{\substack{spin\\orbit}}\ \text{naive}} \underbrace{- \frac{1}{2}\,\frac{e^2}{m_e^2 c^2}\,\frac{1}{R^3}\,\vec{L}\cdot\vec{S}}_{\text{Thomas precession}}$$

Thomas Precession

This is a purely ~~[illegible]~~ relativistic kinematic effect due to the rotating nature of the e^-'s ~~reference~~ instantaneous reference frame.

It stems from the fact that ~~a~~ the result of 2 Lorentz transformations in different directions can only be written as a single Lorentz transformation combined with a rotation.