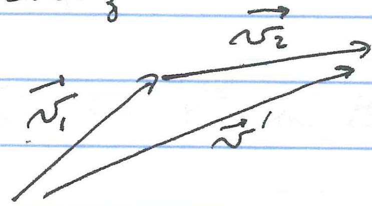
Tuesday, February 5, 2013

#1

Thomas Precession

Relativistic inertial frame rotation $\longrightarrow$ effective spin precession.

origin: 2 Lorentz transformations in different directions can only be written as a single Lorentz transformation combined with a rotation.



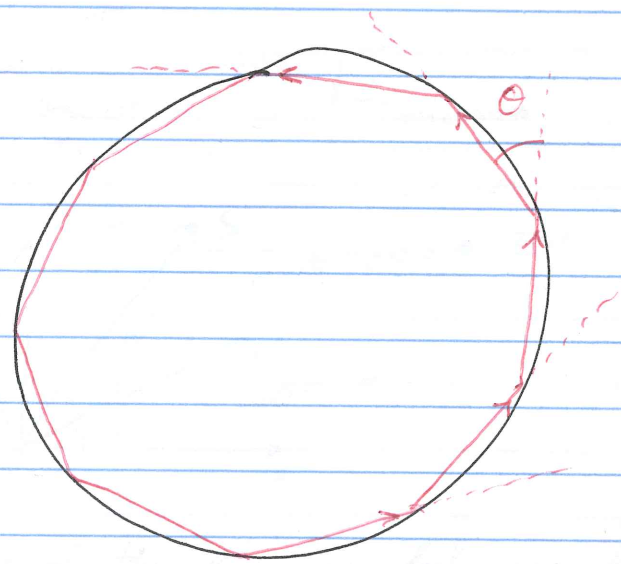
$\vec{v}\,'$ = use relativistic velocity addition

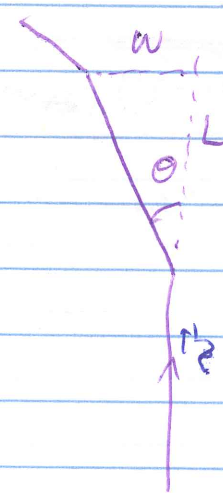
Gallilean relativity	Lorentzian Relativity
$\Gamma(\vec{v}_1)\,\Gamma(\vec{v}_2) = \Gamma(\vec{v}_1+\vec{v}_2)$	$\Lambda(\vec{v}_1)\Lambda(\vec{v}_2) \neq \Lambda(\vec{v}_1+\vec{v}_2)$ $= \Lambda(\vec{v}\,')\ \cancel{R(\theta)}\ R(\theta(\vec{v}_1,\vec{v}_2))$

Simple derivation (for full derivation see Jackson chpt 11.8)

to Smoot & Purcell

Consider an e^- in circular motion, which we approximate as an ~~[illegible]~~ N-sided polygon trajectory:





In the Lab frame: $\theta = \dfrac{2\pi}{N}$

In the frame of the e^-: $\theta' \simeq \tan\theta' = \dfrac{w'}{L'}$ for N large, θ small

~~mostly~~ $\perp$

length contraction $\rightarrow$ ~~mostly~~ $\parallel$

$$\simeq \frac{w}{L/\gamma} = \gamma\frac{w}{L} = \gamma\theta$$

thus $\theta' = \gamma\theta = \gamma\dfrac{2\pi}{N}$

After a full cycle in the lab frame: $\Theta_{Total} = \frac{N 2\pi}{N} = 2\pi$

↳ but ~~for the e~~ in the e⁻'s frame:

$$\Theta'_{TOTAL} = N \gamma \frac{2\pi}{N} = \gamma 2\pi$$

The total angle difference ~~to the~~ is thus

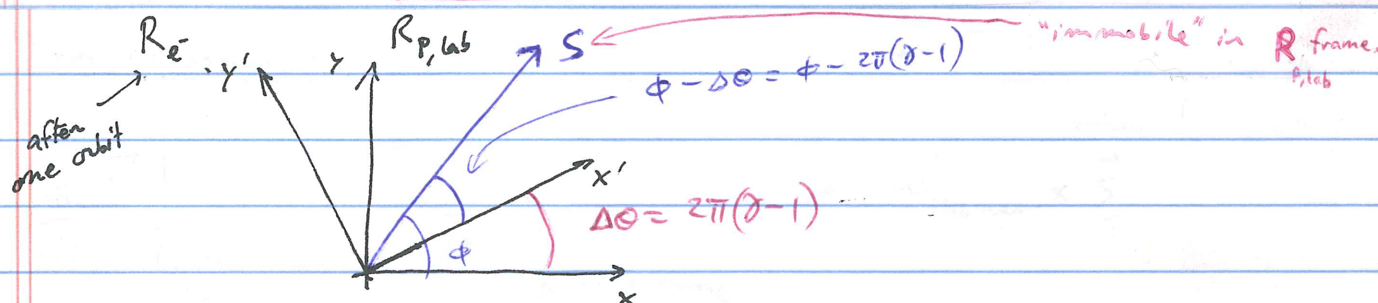
$$\Delta\Theta = \Theta'_{total} - \Theta_{total} = 2\pi(\gamma - 1)$$

In the proton/lab frame one orbit takes a time $T = \gamma \tau$ ← proper time

$$\omega_{Thomas} = \frac{\Delta\Theta}{T} = \frac{2\pi}{T}(\gamma-1) = \omega_{orbit}(\gamma-1)$$

⚠ the orbiting motion of the e⁻ does not have an associated rotating reference frame. The rotation ⇒ of the e⁻ reference frame is the Thomas rotation.

$$\boxed{\omega_{thomas} = \omega_{orbit}(\gamma - 1)}$$



In the frame of the e⁻, it looks as though $\vec{S}$ is precessing clockwise (despite counterclockwise e⁻ motion)

thus in frame of e⁻, we have for $\vec{\omega}'_{thomas} = -(\gamma-1)\omega_{orbit}\hat{z}$

$$\left.\frac{dS}{dt}\right|_{e^- \text{frame}} = \vec{\omega}'_{thomas} \times \vec{S} + \vec{M}_s \times \vec{B}_{eff}$$

Torque $= \frac{d\vec{L}}{dt} = \frac{d\vec{S}}{dt} = \vec{M}_s \times \vec{B}_{eff}$

$$= -\vec{S} \times \vec{\omega}'_{thomas} + \vec{S} \times \frac{g q_e}{2 m_e} \vec{B}_{eff}$$

#3

$\Rightarrow$ the Thomas precession just looks like it is due to some additional fictitious magnetic field (in e^- frame)

we can include the Thomas precession with an interaction Hamiltonian:

$$H_{thomas} = +\vec{S}\cdot\vec{\omega}_{Thomas}$$

$$H_{B_{eff}} = -g\frac{q_e}{2m_e}\vec{S}\cdot\vec{B} = -\vec{M}_s\cdot\vec{B} = \frac{e^2}{m_e^2c^2}\frac{1}{R^3}\vec{L}\cdot\vec{S}$$

note: $\omega_{thomas} = \omega_{orbit}(\gamma - 1)$

for circular motion:

$$T = \frac{2\pi r}{v} \Rightarrow f = \frac{v}{2\pi r} \Rightarrow \omega_{orbit} = \frac{v}{r}$$

$$a = \frac{v^2}{r} \Rightarrow \omega_{orbit} = \frac{a}{v}$$

Also:
$$\gamma - 1 = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} - 1 \simeq \frac{1}{1-\frac{1}{2}\left(\frac{v}{c}\right)^2} - 1 \simeq 1 + \frac{1}{2}\left(\frac{v}{c}\right)^2 - 1 \simeq \frac{1}{2}\left(\frac{v}{c}\right)^2$$

$$= \frac{a}{v}\,\frac{1}{2}\left(\frac{v}{c}\right)^2 = \frac{1}{2}\frac{av}{c^2}$$

we can generalize locally circular motion to

p/lab frame quantities

$$\vec{\omega}'_{thomas} = -\frac{1}{2}\frac{1}{c^2}\vec{v}\times\vec{a}\qquad \frac{\vec{F}}{m_e}$$

$$= -\frac{1}{2}\frac{1}{c^2}\frac{m_e\vec{v}}{m_e}\times\frac{q_e\vec{E}}{m_e}$$

$$= -\frac{1}{2}\frac{1}{m_e^2c^2}\vec{p}\times\underbrace{\frac{q_e|q_e|}{4\pi\varepsilon_0}}_{-e^2}\underbrace{\frac{\hat{r}}{r^2}}_{\frac{\vec{r}}{r^3}}$$

$$= +\frac{1}{2}\frac{e^2}{m_e^2c^2}\frac{1}{r^3}\vec{p}\times\vec{r}$$

$$= -\frac{1}{2}\frac{e^2}{m_e^2c^2}\frac{1}{r^3}\vec{\ell}$$

$$\Rightarrow H_{thomas} = -\frac{1}{2}\frac{e^2}{m_e^2c^2}\frac{1}{R^3}\vec{L}\cdot\vec{S}$$

Thus

$$H_{\substack{\text{Spin}\\ \text{-orbit}}} = H_{B_{eff}} + H_{\text{Thomas}}$$

$$= \frac{e^2}{m_e^2 c^2} \frac{1}{R^3} \vec{L}\cdot\vec{S} \quad - \frac{1}{2} \frac{e^2}{m_e^2 c^2} \frac{1}{R^3} \vec{L}\cdot\vec{S}$$

$$\Rightarrow \boxed{H_{\text{Spin-orbit}} = + \frac{1}{2} \frac{e^2}{m_e^2 c^2} \frac{1}{R^3} \vec{L}\cdot\vec{S}}$$

Spin-orbit coupling in other domains of physics

Nuclear physics / structure:

B_{eff} contribution to $H_{\substack{\text{spin-orbit}\\ \text{nuclear}}}$ is negligible

Thomas precession term dominates $H_{\substack{\text{spin-orbit}\\ \text{nuclear}}}$ → negative
($\vec{F}$ = strong nuclear force)

↳ inversion of nuclear level doublets

Solid state physics:

- Spin-orbit interaction is frequently of the form $H_{so} \propto \vec{p}\cdot\vec{S}$, $\vec{k}\cdot\vec{S}$

- Spin-orbit interaction is responsible for a number of novel solid state phenomena, notably conductivity of surface modes in ~~a~~ topological insulators.

Fine Structure : Hydrogen with spin-orbit coupling

The Hamiltonian for H with spin-orbit coupling is

$$H = H_0 + H_{SO}$$

$$= \underbrace{\frac{P^2}{2m_e} - \frac{e^2}{R}}_{H_0} + \underbrace{\frac{1}{2}\frac{e^2}{m_e^2c^2}\frac{1}{R^3}\vec{L}\cdot\vec{S}}_{H_{SO} = W}$$

We will treat H_{SO} perturbatively

$\vec{L}\cdot\vec{S}$

Relative size of H_{SO} : $\dfrac{\text{"}\langle H_{SO}\rangle\text{"}}{\text{"}\langle H_0\rangle\text{"}} = \dfrac{\cancel{\frac{1}{2}}\cancel{\frac{e^2}{m_e^2c^2}}\frac{1}{a_0^{\cancel{3}2}}\hbar^2}{\cancel{\frac{e^2}{2a_0}}} = \dfrac{\hbar^2}{m_e^2c^2a_0^2}$

$a_0 = \dfrac{\hbar^2}{m_e e^2}$

$$= \frac{\cancel{\hbar^2}}{\cancel{m_e^2}c^2}\frac{\cancel{m_e^2}e^4}{\hbar^{\cancel{4}2}} = \frac{e^4}{\hbar^2c^2} = \alpha^2$$

$$= \left(\frac{1}{137}\right)^2$$

$$\sim 10^{-4}$$

The simplest basis with spin is

$|\psi\rangle = |n, l, m_l\rangle|m_s\rangle$ $\quad$ $|\uparrow\rangle$ or $|\downarrow\rangle$ in $S^2 S_z$ eigenbasis ~~basis~~

$$\langle r|\psi\rangle = R_{nl}(r)\, Y_l^{m_l}(\theta,\phi)\, |m_s\rangle$$

We note the energy levels of H_0 are further degenerate when spin is include

$$H_0|n, l, m_l\rangle \begin{matrix}|\uparrow\rangle \\ |\downarrow\rangle\end{matrix} = E_n |n, l, m_l\rangle \begin{matrix}|\uparrow\rangle \\ |\downarrow\rangle\end{matrix}$$

$\Rightarrow$ we must use degenerate perturbation theory and make a zero-order correction to the basis vectors

compute all the matrix elements!

options: 1. diagonalize H_{so} by brute force in degenerate subspace.

2. diagonalize H_{so} by cleverness

commute

$$\vec{J}^2 = (\vec{L}+\vec{S})^2 = \vec{L}^2 + 2\vec{L}\cdot\vec{S} + \vec{S}^2$$

Stopped here

option 2: $H_{so} = \frac{1}{2}\frac{e^2}{m_e^2c^2}\frac{1}{R^3}\vec{L}\cdot\vec{S} = \frac{1}{2}\frac{e^2}{m_e^2c^2}\frac{1}{R^3}\left[\frac{1}{2}(\vec{J}^2-\vec{L}^2-\vec{S}^2)\right]$

This suggest the following (eigenbasis): $j = \begin{cases} l+\frac{1}{2} \\ l-\frac{1}{2} \end{cases}$

$$|\psi\rangle' = |n\rangle_l\,|J, m_J\rangle$$

$$\vec{J}^2|\psi\rangle' = \hbar^2 j(j+1)|\psi\rangle' \qquad J_z|\psi\rangle' = \hbar m_J|\psi'\rangle$$

$$\vec{L}^2|\psi'\rangle' = \hbar^2 l(l+1)|\psi\rangle' \qquad L_z|\psi'\rangle = ?$$

$$\vec{S}^2|\psi'\rangle' = \hbar^2 \underbrace{s(s+1)}_{3/4}|\psi\rangle' \qquad S_z|\psi'\rangle = ?$$

$|\psi'\rangle$ is not an eigenstate of L_z or S_z.

1st order energy correction

$$\langle\psi'|H_{so}|\psi'\rangle = \frac{1}{4}\frac{e^2}{m_e^2c^2}\,{}_l\langle n\,l|\frac{1}{R^3}|n\,l\rangle_l\left[\frac{\hbar^2}{2}\,j(j+1)-l(l+1)-3/4\right]$$

$$= \frac{1}{2}\frac{\hbar^2}{4}\frac{e^2}{m_e^2c^2}\left[j(j+1)-l(l+1)-3/4\right]\int_0^\infty r^2dr\,\frac{1}{r^3}|R_{nl}(r)|^2$$

for an s-state: $l=0 \Rightarrow j = 0+\frac{1}{2} = \frac{1}{2}$

$\hookrightarrow j(j+1) - l(l+1) - 3/4$

$= 3/4 - 0 - 3/4 = 0$

$\langle\psi'|H_{so}|\psi'\rangle = 0 \Rightarrow$ no correction to the energy.