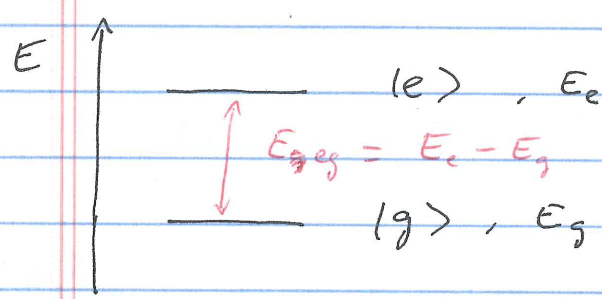
Tuesday, February 26, 2013

## Dynamics of 2-level systems

We consider a generic 2-level quantum mechanical system with 2 distinct eigenstates and eigenenergies




$$\Longleftrightarrow H = \begin{array}{c|cc} & |g\rangle & |e\rangle \\ \hline \langle g| & E_g & 0 \\ \langle e| & 0 & E_e \end{array}$$

## (I) Basic time-dependence

what happens when you let an arbitrary state evolve in time?

Easy: $|\psi\rangle = \alpha|g\rangle + \beta|e\rangle \Rightarrow |\psi(t)\rangle = \alpha e^{-i\frac{E_g}{\hbar}t}|g\rangle + \beta e^{-i\frac{E_e}{\hbar}t}|e\rangle$

$$= \underbrace{e^{-i\frac{E_g}{\hbar}t}}_{\text{overall phase}}\Big[\alpha|g\rangle + \beta \underbrace{e^{-i\frac{(E_e-E_g)t}{\hbar}}}_{\omega_{eg}=\frac{E_{eg}}{\hbar}}|e\rangle\Big]$$

overall phase ↳ not important | relative phase

$$\left.\begin{aligned} P(|g\rangle)_{|\psi(t)\rangle} &= |\langle g|\psi(t)\rangle|^2 = |\alpha|^2 \\ P(|e\rangle)_{|\psi(t)\rangle} &= |\langle e|\psi(t)\rangle|^2 = |\beta|^2 \end{aligned}\right\} \text{no measureable time-dependence}$$

<u>more interesting:</u>

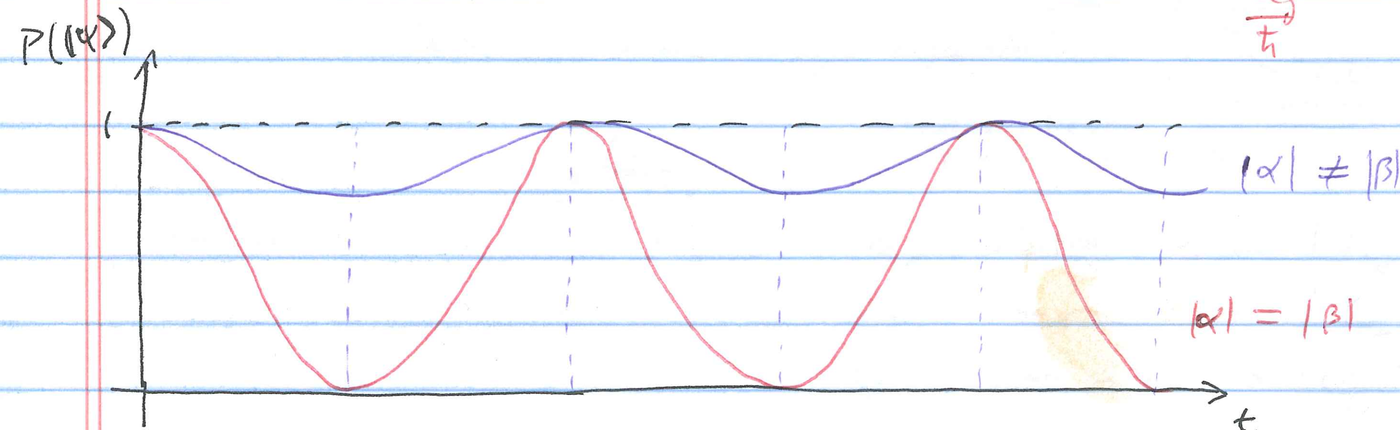
what is the probability to remain in the original state

$$P(|\psi\rangle)_{|\psi(t)\rangle} = |\langle\psi|\psi(t)\rangle|^2$$

$$= \left|\left(\alpha^*\langle g| + \beta^*\langle e|\right)\left(\alpha e^{-i\frac{E_g}{\hbar}t}|g\rangle + \beta e^{-i\frac{E_e}{\hbar}t}|e\rangle\right)\right|^2$$

(some algebra)

$$= \frac{1}{2}\left[1+\left(|\alpha|^2-|\beta|^2\right)^2\right] + \frac{1}{2}\left[1-\left(|\alpha|^2-|\beta|^2\right)^2\right]\cos(\underbrace{\omega_{eg}}_{\frac{E_{eg}}{\hbar}} t)$$



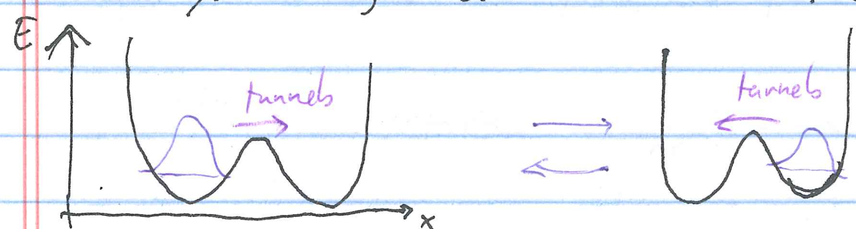

there is a time-dependence if you look another basis

<u>examples:</u>

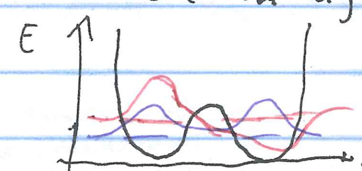
1_ <u>AC Josephson effect</u> (or Josephson oscillations)

consider a double well potential

if you put your particles ($e^-$'s, atoms, etc...) in one well, then you will get oscillation between the two wells




why? measurements are not done in eigenbasis




superconductor: SC – I – SC junction or BEC in a double well

If we consider just the ground state $|\psi_0\rangle$ and $1^{st}$ excited state $|\psi_1\rangle$, then

$$\underbrace{|left\rangle}_{\text{measurement basis}} = \underbrace{|\psi_0\rangle + |\psi_1\rangle}_{\text{energy eigenbasis}} \quad \text{orthogonal}$$

$$|right\rangle = |\psi_0\rangle - |\psi_1\rangle$$

note: $|\psi_0\rangle$ & $|\psi_1\rangle$ can be single particle states or many-body eigenstates.

Rabi Régime (non-interacting particles)

Josephson Régime (weak inter-particle interactions)

## 2 - Neutrino oscillations

Neutrino's come in 3 flavors (+ anti-particles)

$\nu_e$, electron $-\nu \rightarrow |\psi_{\nu_e}\rangle$

$\nu_\mu$ muon $-\nu \rightarrow |\psi_{\nu_\mu}\rangle$

$\nu_\tau$ tau $-\nu \rightarrow |\psi_{\nu_\tau}\rangle$

These are the measurement basis, but not the energy eigenbasis (i.e. mass basis): $|\psi_{\nu_{m_1}}\rangle$, $E \simeq pc + \frac{1}{2} m_1^2 \frac{c^3}{p}$

$|\psi_{\nu_{m_2}}\rangle$, $E \simeq pc + \frac{1}{2} m_2^2 \frac{c^3}{p}$

$|\psi_{\nu_{m_3}}\rangle$, $E \simeq pc + \frac{1}{2} m_3^2 \frac{c^3}{p}$

Relativistic Energy: $E = \sqrt{p^2c^2 + m_0^2c^4} \simeq pc + \frac{1}{2} m_0^2 \frac{c^3}{p}$

(for neutrinos $E_{kinetic} \ggg E_{rest} = m_0 c^2$)

The time evolution of the neutrino wavefunction is determined in the mass/energy eigenbasis

$$|\psi(t)\rangle = \alpha e^{-i\frac{E_1 t}{\hbar}} |\psi_{\nu_{m_1}}\rangle + \beta e^{-i\frac{E_2 t}{\hbar}} |\psi_{\nu_{m_2}}\rangle + \gamma e^{-i\frac{E_3 t}{\hbar}} |\psi_{\nu_{m_3}}\rangle$$

with for example $|\psi_{\nu_e}\rangle = \alpha_e |\psi_{\nu_{m_1}}\rangle + \beta_e |\psi_{\nu_{m_2}}\rangle + \gamma_e |\psi_{\nu_{m_3}}\rangle$

change of basis matrix

= Pontecorvo-Maki-Nakagawa-Sakata matrix (P)MNS matrix

We see oscillation because measurement and mass basis are different

↳ one can only see oscillations proportional to ~~mass differences~~ the difference of masses squared: $\Delta E \propto \Delta m_{ij}^2 = m_i^2 - m_j^2$

3. Rabi oscillations in the dressed atom picture

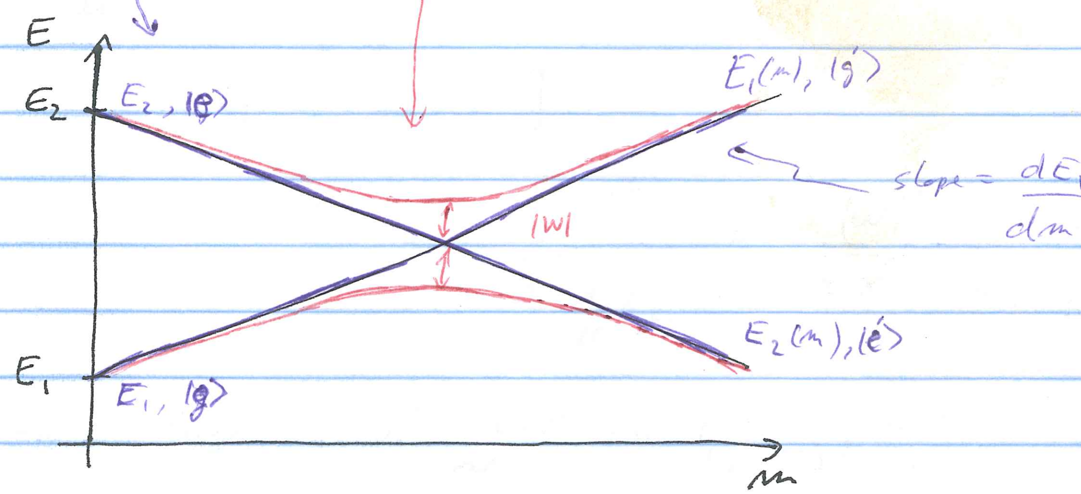
(later today) atom + simplified quantized E-M field.

## (II) Landau - Zener Transitions

We consider a 2-level system with an off-diagonal interaction $W$ (inter-state)

in which the base energies are varied through each other:

$$H = \underbrace{\begin{array}{c|cc} & |g\rangle & |e\rangle \\ \hline \langle g| & E_1(m) & 0 \\ \langle e| & 0 & E_2(m) \end{array}}\; + \begin{bmatrix} 0 & W \\ W^* & 0 \end{bmatrix}$$




**Effect of W**

There is an avoided level crossing, and $E_2(m=0)$ $|e\rangle$ is connected to $E_1(m \to \text{large}), |g'\rangle$, and $E_1(m=0), |g\rangle$ is connected to ~~$E_2$~~ $E_2(m \to \text{large}), |e'\rangle$.

**Qualitative time-dependence**

If $m$ is varied "slowly" in time (e.g. $m = \alpha t$ with $\alpha$ small), then if the system starts in $|g\rangle$ with $E = E_1(m=0)$, then we expect to end up in $|e'\rangle$ with $E = E_2(m = \text{large})$

Likewise, we expect $|e\rangle$ with $E=E_2(m=0) \longrightarrow |g'\rangle$ with $E=E_1$ (m=large)

If $m$ is varied "quickly" (diabatically) in time (e.g. $m=\alpha t$ with $\alpha$ large), then we expect the system to "jump" the no-level crossing gap:

$$E=E_1(m=0),\ |g\rangle \longrightarrow E=E_1(m=\text{large}),\ |g'\rangle$$

$$E=E_2(m=0),\ |e\rangle \longrightarrow E=E_2(m=\text{large}),\ |e'\rangle$$

## Quantitative time-dependence

what is the probability to end up in state $|g'\rangle$ ($m$ large, $m\to+\infty$) given an initial state of $|g\rangle$ ($m\sim 0$, $m\to-\infty$) and an energy ramp $\frac{d}{dt}(E_2-E_1)=\frac{d}{dt}E_{21}=\gamma$?

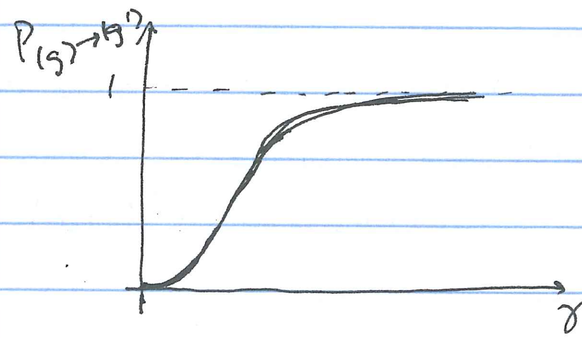
Answer: $$P_{|g\rangle\to|g'\rangle} = \exp\left[-2\pi\Gamma\right]$$

$|g\rangle\to|g'\rangle$ "jump" ~~is called~~ is called a Landau-Zener transition ~~tdiab~~

with $$\Gamma = \frac{|W|^2}{\hbar\cdot\frac{d}{dt}E_{21}} = \frac{|W|^2}{\hbar\gamma} = \frac{[J]^2}{([J]\cdot[s])[J]/[s]} = \text{dimensionless}$$

Fast ramp $\to \gamma$ large $\to \Gamma$ small $\sim 0 \Rightarrow P_{|g\rangle\to|g'\rangle}\sim 1$

Slow ramp $\to \gamma\sim 0 \to \Gamma$ large $\Rightarrow P_{|g\rangle\to|g'\rangle}\sim 0$



## III Rabi flopping (Rabi oscillations)

We consider a 2-level system driven by a sinusoidal (time-dependent) perturbation. ("exactly" solvable)

$$H = \begin{pmatrix} E_g & 0 \\ 0 & E_e \end{pmatrix} + \hbar \begin{pmatrix} 0 & \Omega \\ \Omega^* & 0 \end{pmatrix} \cos(\omega t)$$

Classic example: 2-level atom driven by an electromagnetic wave

Also, NMR: nuclear spin (typically proton) driven by a sinusoidal B-field in a large DC B-field.

also, EPR (electron paramagnetic resonance)

$$\Omega = \frac{\langle g | W | e \rangle}{\hbar}$$

with $W = e\vec{R}\cdot\vec{E}_0 \cos(\omega t)$

(ignore role of B-field → consider only E1 transitions)

What's the time evolution of the system?

The Schrodinger equation is

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = H |\psi\rangle \qquad (1)$$

We will find a solution in the unperturbed basis $\{|g\rangle, |e\rangle\}$

also the measurement basis

$$|\psi(t)\rangle = C_g(t)\, e^{-i\omega_g t} |g\rangle + C_e(t)\, e^{-i\omega_e t} |e\rangle \qquad (2)$$

$E_g/\hbar$ $\quad$ $E_e/\hbar$

non-trivial time dependence

(2) → (1) "plug & chug"

looking at the $|g\rangle$ component

$$i\hbar\left[\frac{d}{dt}c_g(t)\cdot e^{-i\omega_g t} + \cancel{c_g(t)(-i\omega_g)e^{-i\omega_g t}}\right]|g\rangle$$

$$= \left[\cancel{c_g(t)e^{-i\omega_g t}E_g} + c_e(t)e^{-i\omega_e t}\hbar\Omega\cos(\omega t)\right]|g\rangle$$

thus

$$i\hbar\frac{d}{dt}c_g(t) = c_e(t)\,\hbar\Omega\cos(\omega t)\,e^{-i\omega_{eg}t} \qquad (3a)$$

$\omega_{eg} = \omega_e - \omega_g$

similarly,

($|e\rangle$ component)

$$i\hbar\frac{d}{dt}c_e(t) = c_g(t)\,\hbar\Omega^*\cos(\omega t)\,e^{+i\omega_{eg}t} \qquad (3b)$$

no approximations so far

<u>Rotating wave approximation</u> (RWA)

(3a) → $$i\hbar\frac{d}{dt}c_g(t) = c_e(t)\hbar\Omega\left[\frac{e^{i\omega t}+e^{-i\omega t}}{2}\right]e^{-i\omega_{eg}t}$$

$$= \frac{c_e(t)\hbar\Omega}{2}\left[\underbrace{e^{i(\omega-\omega_{eg})t}}_{\text{slow oscillations for } \omega\sim\omega_{eg}} + \underbrace{e^{-i(\omega+\omega_{eg})t}}_{\text{very fast oscillations}}\right]$$

↳ ignore (RWA) (averages to "zero")

<u>partial justification</u>: if $c_e(t)\sim cst$

integrate $\dot{c}_g(t) \Rightarrow i\,c_g(t) \simeq c_e\frac{\Omega}{2}\left[\frac{e^{i(\omega-\omega_{eg})t}}{i(\omega-\omega_{eg})} + \frac{e^{-i(\omega+\omega_{eg})t}}{-i(\omega+\omega_{eg})}\right]$

very small

stopped here