Thursday, February 28, 2013

Rabi oscillations (continued)

E_e —— $|e\rangle$

E_g —o— $|g\rangle$

$+ \hbar \begin{pmatrix} 0 & \Omega \\ \Omega^* & 0 \end{pmatrix} \cos(\omega t)$ #1

thus, we have (after the Rotating wave approximation)

$$\begin{cases} i\hbar \dfrac{d}{dt} c_g(t) = c_e(t) \dfrac{\hbar\Omega}{2} e^{-i(\omega_{eg}-\omega)t} & (4a) \\ \text{similarly,} & \\ i\hbar \dfrac{d}{dt} c_e(t) = c_g(t) \dfrac{\hbar\Omega^*}{2} e^{i(\omega_{eg}-\omega)t} & (4b) \end{cases}$$

we need separate equations for $c_g(t)$ and $c_e(t)$

differentiate $\frac{d}{dt}(4a)$, $\frac{d}{dt}(4b)$ and substitute (4a) & (4b) into resulting equations

$$\begin{cases} \dfrac{d^2}{dt^2} c_g - i\delta \dfrac{d}{dt} c_g + \dfrac{|\Omega|^2}{4} c_g = 0 & (5a) \\ \dfrac{d^2}{dt^2} c_e + i\delta \dfrac{d}{dt} c_e + \dfrac{|\Omega|^2}{4} c_e = 0 & (5b) \end{cases}$$

with $\delta = \omega - \omega_{eg}$

<u>Initial conditions</u>: system initially in ground state

$c_g(t=0) = 1$, $\frac{d}{dt} c_g(t=0) = 0$

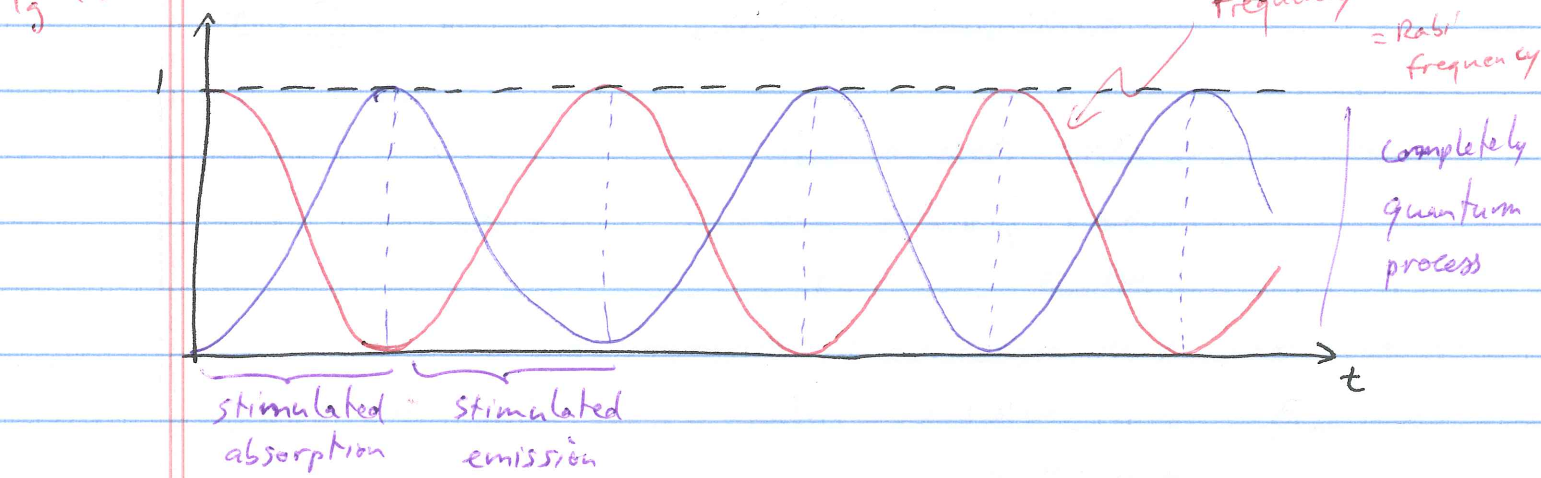
$c_e(t=0) = 0$, $\frac{d}{dt} c_e(t=0) = 0$

solution (simple but tedious algebra)

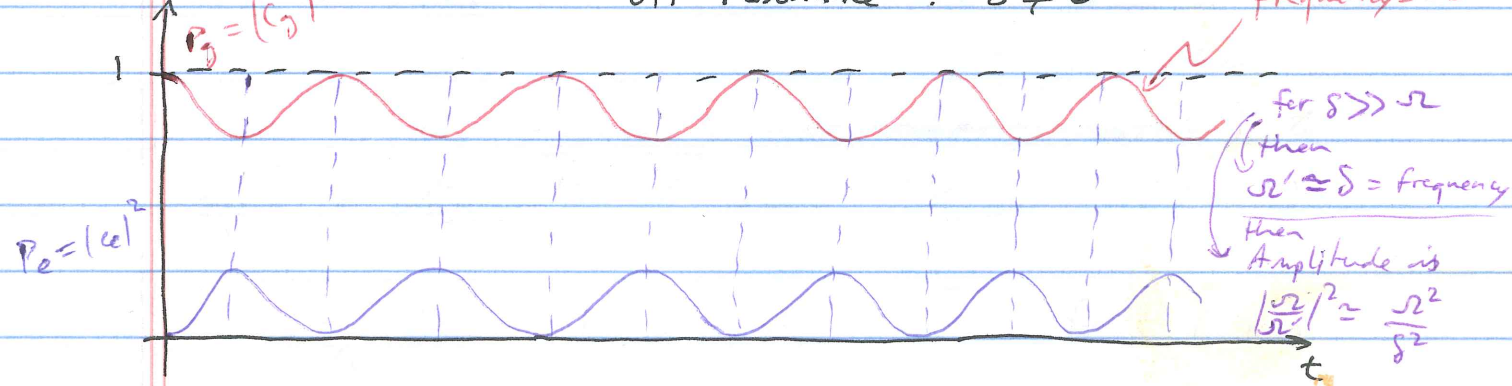
$$c_g(t) = \left[\cos\left(\frac{\Omega' t}{2}\right) - \frac{i\delta}{\Omega'} \sin\left(\frac{\Omega' t}{2}\right)\right] e^{+i\frac{\delta}{2}t} \qquad (6a)$$

$$c_e(t) = -\frac{i\Omega}{\Omega'} \sin\left(\frac{\Omega' t}{2}\right) e^{-i\frac{\delta}{2}t} \qquad (6b)$$

$\delta = \omega - \omega_{eg}$, Ω = Rabi frequency, $\Omega' = \sqrt{\Omega^2 + \delta^2}$ = generalized Rabi frequency "on resonance": $\delta = 0$



"off-resonance": $\delta \neq 0$



~~Bloch Sphere~~ [show 5-level Rabi flopping movie]

* Time-dependent perturbation theory

Consider a Hamiltonian H_0 ~~subject~~ with eigenstates and eigenenergies: $\{E_n, |\varphi_n\rangle\}$ $n = 0, 1, 2, \ldots$

$(H_0|\varphi_n\rangle = E_n|\varphi_n\rangle)$

We apply a ~~perturbation~~ time-dependent perturbation $W(t)$ at time $t \geq 0$ with $|\underbrace{\langle\varphi_n|W|\varphi_m\rangle}_{W_{nm}(t)}| \ll |E_u - E_v|$ for $\forall\, n, m, u, v$

If at time $t<0$, the system is in an eigenstate $|\psi(t<0)\rangle = |\varphi_i\rangle$, then what is the probability (~~transition probability~~) to make a transition to state $|\varphi_j\rangle$ $(j \neq i)$:

$$P_{ij}(t) = |\langle\varphi_j|\psi(t)\rangle|^2 \quad \text{for } t \geq 0$$

As with the Rabi problem, we write

$$|\psi(t)\rangle = \sum_n c_n(t) e^{-i\omega_n t} |\varphi_n\rangle \quad \text{where } \omega_n = \frac{E_n}{\hbar}$$

note: $P_{ij}(t) = |c_j(t)|^2$

We write the perturbation as $W(t) = \lambda \hat{W}(t)$ with $\lambda \ll 1$

The time-dependent Schrodinger equation is

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = [H_0 + \lambda\hat{W}(t)]|\psi(t)\rangle$$

$\langle\varphi_k|$

$$i\hbar\left[\frac{d}{dt}c_k(t)\cdot e^{-i\omega_k t} + \cancel{c_k(t)(-i\omega_k)e^{-i\omega_k t}}\right]$$

$$= \cancel{E_k c_k(t) e^{-i\omega_k t}} + \lambda\sum_n \hat{W}_{kn}(t) c_n(t) e^{-i\omega_n t}$$

$$\Leftrightarrow \quad i\hbar \frac{d}{dt} C_k(t) = \lambda \sum_n \hat{W}_{kn}(t) C_n(t) e^{i\omega_{kn} t}$$

with $\omega_{kn} = \omega_k - \omega_n$

If we can solve this system of equations, then

$$C_k(t) = f(\lambda, W_{nm}(t), \omega_{nm}; t)$$

so we propose a Taylor expansion in λ for $C_k(t)$:

$$C_k(t) = C_k^{(0)}(t) + \lambda C_k^{(1)}(t) + \lambda^2 C_k^{(2)}(t) + \dots$$
$$= \sum_{r=0}^{\infty} \lambda^r C_k^{(r)}(t)$$

thus, we have

$$i\hbar \sum_{r=0}^{\infty} \lambda^r \frac{d}{dt} C_k^{(r)}(t) = \sum_{r=0}^{\infty} \lambda^{r+1} \left(\sum_n \hat{W}_{kn}(t) C_n^{(r)}(t) e^{i\omega_{kn} t} \right)$$

the above expression must be true as we vary λ, so the equality is true each λ^r terms, Thus

r^{th} term (on LHS)

$$\boxed{i\hbar \frac{d}{dt} C_k^{(r)}(t) = \sum_n \hat{W}_{kn}(t) C_n^{(r-1)}(t) e^{i\omega_{kn} t}}$$

[recurrence relation] if you know $C_n^r(t)$ then you can get $C_n^{r+1}(t)$

in the case of $r=0$, then $i\hbar \frac{d}{dt} C_k^{(0)}(t) = 0 \Leftrightarrow C_k^{(0)}(t) = cst$

↳ this the result if $\lambda = 0$ (i.e. no perturbation)

<u>Initial conditions</u>:

for $t<0$ all $\underset{n \neq i}{C_n(t)} = 0$ except $C_i(t) = 1$ [neglect overall phase]

at $t=0$, $W(t) = \lambda \hat{W}(t)$ is turned <u>on</u> $(W(t<0) = 0)$

note: this turn-on can be gradual or abrupt

even if abrupt $\left(\text{i.e. } \begin{array}{l} W(t<0)=0 \\ W(t=0) \neq 0 \end{array} \right)$, the

wave function $|\Psi(t)\rangle$ remains continuous:

$$i\hbar \frac{d}{dt} c_k(t) \Big|_{t=0} = \cancel{H_0 + \hat{W}(t)} \underbrace{\lambda \sum_n \hat{W}_{kn}(t=0)\, c_n(t)\, e^{i\omega_{kn} t}\Big|_{t=0}}_{\text{stays finite}}$$

so $\underset{k \neq i}{c_k(t=0)} = 0$ and $c_i(t=0) = 1$

or $c_k(t=0) = \delta_{ki}$

$$\Rightarrow \begin{cases} c_k^{(0)}(t=0) = \delta_{ki} \\ c_k^{(r)}(t=0) = 0 \quad \text{for } r \geq 1 \end{cases}$$

for r=0: $c_k^{(0)}(t) = cst$, so $\boxed{c_k^{(0)}(t) = \delta_{ki}}$

note: the derivation has not enforced normalization.

for r=1: then $i\hbar \frac{d}{dt} c_k^{(1)}(t) = \sum_n \hat{W}_{kn}(t)\, \underbrace{c_n^{(0)}(t)}_{\delta_{ni}}\, e^{i\omega_{kn} t}$

$$\Rightarrow i\hbar \frac{d}{dt} c_k^{(1)}(t) = \hat{W}_{ki}(t)\, e^{i\omega_{ki} t}$$

$$\Rightarrow c_k^{(1)}(t) = \frac{1}{i\hbar} \int_0^t \hat{W}_{ki}(t')\, e^{i\omega_{ki} t'}\, dt'$$

To ~~~~ 1^{st} order with $k \neq i$

$$C_k(t) \simeq \cancel{C_k^{(0)}(t)} + \lambda C_k^{(1)}(t) \simeq \frac{1}{i\hbar}\int_0^t \langle \varphi_k | W(t') | \varphi_i \rangle e^{i\omega_{ki}t'} dt'$$

$$\boxed{\Rightarrow \underset{k \neq i}{C_k(t)} \simeq \frac{1}{i\hbar}\int_0^t \langle \varphi_k | W(t') | \varphi_i \rangle e^{i\omega_{ki}t'} dt'}$$

and

$$\boxed{\begin{aligned} P_{ik}(t) &= |\langle \varphi_i | \psi(t) \rangle|^2 = \left| \underset{k\neq i}{C_k(t)} \right|^2 \\ &= \frac{1}{\hbar^2}\left| \int_0^t \langle \varphi_k | W(t') | \varphi_i \rangle e^{i\omega_{ki}t'} dt' \right|^2 \end{aligned}}$$

stopped here

$C_k^{(1)}(t) \sim$ Fourier component of $\underbrace{W(t) \text{ seen by } |\varphi_k\rangle}_{\langle \varphi_k | W(t) | \varphi_i \rangle}$

at $\omega = \omega_k - \omega_i = \omega_{ki}$

note 1: the transition from $|\varphi_i\rangle$ to $|\varphi_k\rangle$ corresponds to the absorption of one quantum of energy at $\Delta E = \hbar\omega_{ki}$.

note 2: $C_i(t) \simeq 1 + \lambda C_i^{(1)}(t) \simeq 1 + \frac{1}{i\hbar}\int_0^t \langle \varphi_i | W(t') | \varphi_i \rangle dt'$

$$\Rightarrow P_{ii}(t) \simeq \left| 1 + \frac{1}{i\hbar}\int_0^t \langle \varphi_i | W(t') | \varphi_i \rangle dt' \right|^2$$

alternatively, use the normalization condition: $P_{ii}(t) = 1 - \sum_k P_{ik}(t)$