Tuesday, March 18, 2013

$$P_{ji}(t) = |c_j(t)|^2 = \frac{|\langle \varphi_j | W | \varphi_i \rangle|^2}{\hbar^2} \, 2\pi\hbar t \, \delta(E_j - E_i)$$ #1

for $t \to$ large

Recall: Thus a step function perturbation tends to drive transitions only between states of equal energy (for $t \to$ large

~~conclusion~~

Transitions / Coupling to a continuous energy spectrum

If instead of a final state $|\varphi_j\rangle$, E_j there is continuum of final states $|\varphi(\alpha)\rangle$, $E(\alpha)$ with a density of states $\rho(E)\,dE$, then we are interested in calculating

$|\varphi(\alpha)\rangle$ are all orthogonal

$$\delta P_{ji} \to \delta P_{\delta\alpha, i} = \int_{\alpha \in \delta\alpha \text{ interval}} d\alpha \, |\langle \varphi(\alpha) | \psi(t) \rangle|^2$$

if we can associate the continuous index α with an energy, then we have

$$\delta P_{\delta\alpha, i} \to \delta P_{\delta E, i} = \int_{E \in \delta E} dE \, \rho(E) \, |\langle \varphi(E) | \psi(t) \rangle|^2$$

for a step function perturbation $W(t)$, we have

$$|\langle \varphi(E) | \psi(t) \rangle|^2 = \frac{|\langle \varphi(E) | W | \varphi_i \rangle|^2}{\hbar^2} \underbrace{\left(\frac{\sin\left[\left(\frac{E - E_i}{2\hbar}\right) t\right]}{\left(\frac{E - E_i}{2\hbar}\right)} \right)^2}_{\simeq 2\pi\hbar t \, \delta(E_j - E_i)}$$

$\delta P_{\delta E, i}$ integral is easy!

$$\delta P_{\delta E, i} = \begin{cases} \frac{2\pi}{\hbar} |\langle \varphi_{(E)} | W | \varphi_i \rangle|^2 t \, \rho(E = E_i) & \text{for } E_i \in \delta E \\ 0 & \text{for } E_i \notin \delta E \end{cases}$$

For $E_i \in \delta E$, the probability of a transition increases linearly with time

$$\Rightarrow \quad \delta P_{\delta E, i} = \Gamma_{i \to \delta E} \, t$$

where $\Gamma_{i \to \delta E}$ is the transition rate

$\Gamma_{i \to \delta E}$ = probability per unit time that a state $|\varphi_i\rangle$ will make a transition to another continuum state $|\varphi_f\rangle$ with energy ~~E_f~~ $E_i \in \delta E$ (i.e. $E_i = E_f$)

$$= \frac{2\pi}{\hbar} |\langle \varphi_f(E) | W | \varphi_i \rangle|^2 \rho(E_f = E_i)$$

Fermi's Golden Rule.

Decay of a discrete state coupled to a continuum

(see also Wigner–Weisskopf theory)

Fermi Golden rule is valid only for short times and tells us how the continuum states become populated.

Instead, we want an expression for $P_{ii}(t)$, the probability to remain in the initial state $|\varphi_i\rangle$ for $t \geq 0$

According, to the Fermi Golden rule

$$\Gamma_{i\to f, E_f = E_i} = \frac{2\pi}{\hbar} |\langle \varphi_f(E)|W|\varphi_i\rangle|^2 \rho(E_f = E_i)$$

we define Γ = probability per unit time to make a transition to any state, such that $E_f = E_i$

$$= \frac{2\pi}{\hbar}\int d\alpha\, |\langle \varphi_{f\alpha}(E)|W|\varphi_i\rangle|^2 \rho(E_{f\alpha} = E_i)$$

thus $P_{ii}(t) = 1 - \Gamma t$ for t short (i.e. $t \ll \frac{1}{\Gamma}$)

$(P_{ii}(t) \sim 1)$

we want an expression for all $t \geq 0$.

the state of the system can be written as

$$|\psi(t)\rangle = c_i(t)\, e^{-i\frac{E_i t}{\hbar}} |\varphi_i\rangle + \int d\alpha\, c_\alpha(t)\, e^{-i\frac{E_\alpha t}{\hbar}} |\varphi_\alpha\rangle$$

final states with energy $E_f = E_\alpha = E_i$

note: we choose the perturbing coupling, such that

$\langle\varphi_i|W|\varphi_i\rangle = 0$

$\langle\varphi_\alpha|W|\varphi_{\alpha'}\rangle = 0$ and $\langle\varphi_\alpha|W|\varphi_\alpha\rangle = 0$

$\langle\varphi_i|W|\varphi_\alpha\rangle \neq 0$

the Schrodinger equation is

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = \left[H_0 + W\right]|\psi(t)\rangle$$

$\langle\varphi_i|$

$$i\hbar\left[\frac{d}{dt}c_i(t)\cdot e^{-i\frac{E_i t}{\hbar}} + \cancel{c_i(t)\cdot\left(-i\frac{E_i}{\hbar}\right)e^{-i\frac{E_i t}{\hbar}}}\right]$$

$$= \cancel{c_i(t)\, e^{-i\frac{E_i t}{\hbar}} E_i} + \int d\alpha\, c_\alpha(t)\, e^{-i\frac{E_\alpha t}{\hbar}} \langle\varphi_i|W|\varphi_\alpha\rangle$$

$$\Rightarrow \boxed{i\hbar\frac{d}{dt}c_i(t) = \int d\alpha\, c_\alpha(t)\, e^{i\frac{(E_i - E_\alpha)t}{\hbar}} \langle\varphi_i|W|\varphi_\alpha\rangle} \quad (1)$$

apply $\langle\varphi_\alpha|$

similarly,

$$i\hbar\left[\frac{d}{dt}c_\alpha(t)\cdot e^{-i\frac{E_\alpha t}{\hbar}} + \cancel{c_\alpha(t)\left(-i\frac{E_\alpha}{\hbar}\right)e^{-i\frac{E_\alpha t}{\hbar}}}\right]$$

$$= \cancel{c_\alpha(t)\, e^{-i\frac{E_\alpha t}{\hbar}} E_\alpha} + c_i(t)\, e^{-i\frac{E_i t}{\hbar}} \langle\varphi_\alpha|W|\varphi_i\rangle$$

$$\Rightarrow \boxed{i\hbar \frac{d}{dt} C_\alpha(t) = C_i(t)\, e^{-i\frac{(E_i - E_\alpha)t}{\hbar}} \langle \varphi_\alpha | W | \varphi_i \rangle} \quad (2)$$

initial conditions $\begin{cases} C_i(t=0) = 1 \\ C_\alpha(t=0) = 0 \end{cases}$

~~equation 1 becomes~~: Integrate equation 2:

$$C_\alpha(t) = \frac{1}{i\hbar} \int_0^t \langle \varphi_\alpha | W | \varphi_i \rangle\, e^{-i\frac{(E_i - E_\alpha)t'}{\hbar}}\, C_i(t')\, dt'$$

and plug into (1):

$$\frac{d}{dt} C_i(t) = -\frac{1}{\hbar^2} \int d\alpha \int_0^t dt' \Big[\underbrace{e^{+i\frac{(E_i - E_\alpha)}{\hbar}(t-t')}}_{\text{strongly peaked around } \begin{cases} E_i = E_\alpha \\ t = t' \end{cases}} C_i(t')\, |\langle \varphi_\alpha | W | \varphi_i \rangle|^2 \Big]$$

integro-differential equation $\rightarrow$ hard to solve exactly

since only get significant contributions to the integral for $t \simeq t'$, we make the approximation $C_i(t') \simeq C_i(t)$. In this case

$$\frac{d}{dt} C_i(t) = -\frac{C_i(t)}{\hbar^2} \int d\alpha \int_0^t dt' \Big[e^{i\frac{(E_i - E_\alpha)}{\hbar}(t-t')} |\langle \varphi_\alpha | W | \varphi_i \rangle|^2 \Big]$$

$$= -\frac{C_i(t)}{\hbar^2} \int_0^\infty dE_\alpha \int_0^t dt' \Big[\rho(E_\alpha)\, e^{i\frac{(E_i - E_\alpha)}{\hbar}(t-t')} |\langle \varphi_\alpha | W | \varphi_i \rangle|^2 \Big]$$

substitution: $\tau = t - t'$

$$\frac{d}{dt} c_i(t) = -\frac{c_i(t)}{\hbar^2}\int_0^\infty dE_\alpha \int_0^t d\tau \left[\rho(E_\alpha)\, e^{i\frac{(E_i-E_\alpha)}{\hbar}\tau}\, |\langle\varphi_\alpha|W|\varphi_i\rangle|^2\right]$$

$$= \hbar\pi\,\delta(E_i - E_\alpha) + \hbar i\, P\left(\frac{1}{E_i-E_\alpha}\right)$$

Cauchy principal part

for $t \to \infty$

$$P\left(\frac{1}{x}\right) = \lim_{\eta\to 0}\int_{-\infty}^{-\eta} + \int_{+\eta}^{\infty} \frac{1}{x}\,dx$$

$$\Rightarrow\ = \underbrace{-\frac{c_i(t)}{\hbar^2}\left[\hbar\pi\int \rho(E_\alpha = E_i)\,|\langle\varphi_\alpha|W|\varphi_i\rangle|^2\, d\alpha\right.}_{\Gamma/2} + \hbar i\, \underbrace{\left. P\int_0^\infty dE_\alpha\, \frac{\rho(E_\alpha)|\langle\varphi_\alpha|W|\varphi_i\rangle|^2}{E_i-E_\alpha}\right]}_{\delta E}$$

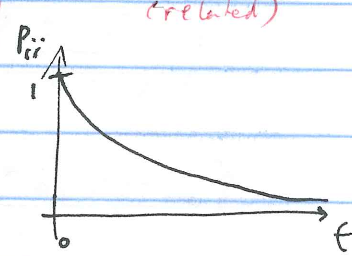
some energy shift (can generally be neglected)

$$\Rightarrow \frac{d}{dt} c_i(t) = -c_i(t)\left[\frac{\Gamma}{2} + i\frac{\delta E}{\hbar}\right]$$

$$\Rightarrow c_i(t) = e^{-\frac{\Gamma}{2}t - i\frac{\delta E}{\hbar}t}$$

$$\boxed{\begin{aligned} &\Rightarrow |\varphi_i(t)\rangle = e^{-\Gamma/2\, t - i\left(\frac{E_i+\delta E}{\hbar}\right)t} \\ &\Rightarrow P_{ii}(t) = e^{-\Gamma t} \end{aligned}}$$

corresponds to an energy shift ("Lamb shift" related)



note: Population decay twice as fast as wave function

note: δE is what we expect from 2nd order time independent perturbation theory!!!