Thursday, March 21, 2013

Discrete Symmetries

Spatial Inversion Symmetry or Parity Symmetry

Parity operation:

Classical: Parity transformation $\begin{cases} x \longrightarrow -x \\ y \longrightarrow -y \\ z \longrightarrow -z \end{cases}$ (see Jackson 6.11 2nd ed.)

examples:

position vector: $\vec{r} \xrightarrow{P} -\vec{r}$ (parity odd)

velocity: $\vec{v} = \frac{d\vec{r}}{dt} \xrightarrow{P} -\frac{d\vec{r}}{dt} = -\vec{v}$ (odd)

momentum: $\vec{p} = m\vec{v} \xrightarrow{P} -m\vec{v} = -\vec{p}$ (odd)

angular momentum: $\vec{L} = \vec{r}\times\vec{p} \xrightarrow{P} (-\vec{r})\times(-\vec{p}) = \vec{L}$ (even)

Force: $\vec{F} = m\vec{a} = m\frac{d^2\vec{r}}{dt^2} \xrightarrow{P} m\left(-\frac{d^2\vec{r}}{dt^2}\right) = -\vec{F}$ (odd)

Energy: $E = \frac{1}{2}mv^2 \xrightarrow{P} \frac{1}{2}m(-v)^2 = E$ (even)

Electric & magnetic fields: consider Lorentz force law

$$\underset{\text{odd}}{F} = q\vec{E} + q\underset{\text{odd}}{\vec{v}}\times\vec{B}$$

$$\Rightarrow \begin{cases} \vec{E} \xrightarrow{P} -\vec{E} & \text{odd} \\ \vec{B} \xrightarrow{P} \vec{B} & \text{even} \end{cases}$$

terminology: vector quantities that are ~~odd under~~

- odd under parity are called "vectors" (or polar vectors)
- even under parity are called "pseudovectors"

note: You cannot accomplish a parity transformation through a series of rotations.

Quantum Mechanics: Parity (transformation) operator Π

$\Pi|\psi\rangle = ?$

we require $\langle\psi|\Pi^{+} X \Pi|\psi\rangle = -\langle\psi|X|\psi\rangle$

$\Rightarrow \quad \Pi^{+} X \Pi = -X$

$\Rightarrow$ Π should be unitary since norm should not change (or ~~expecta~~ probabilities)

$(\langle\psi|\Pi^{+})(\Pi|\psi\rangle) = \langle\psi|\psi\rangle$

$\Rightarrow \quad \Pi^{+}\Pi = 1 \quad \Leftrightarrow \quad \Pi^{+} = \Pi^{-1}$

thus $\quad \Pi\Pi^{+} X \Pi = -\Pi X$

$\Rightarrow \quad X\Pi = -\Pi X \quad \Rightarrow \quad \Pi X + X\Pi = 0$

Consider $|\psi\rangle = |x\rangle$ (position state)

$X\Pi|x\rangle = -\Pi X|x\rangle = -\Pi x|x\rangle$

$= (-x)\Pi|x\rangle$

thus $X(\Pi|x\rangle) = (-x)(\Pi|x\rangle)$

$\Rightarrow \quad \Pi|x\rangle = e^{i\delta}|-x\rangle \equiv |-x\rangle$

by convention $(\delta = 0)$

$$\boxed{\Pi|x\rangle = |-x\rangle}$$

note $\quad \Pi(\Pi|x\rangle) = \Pi|-x\rangle = |x\rangle \Rightarrow \Pi^2 = 1$

$\Rightarrow \quad \Pi = \Pi^{-1} = \Pi^{+}$

$\Rightarrow$ Π is Hermitian

the eigenvalues of Π are ± 1.

~~$|P\rangle$ is an eigenstate of Π~~

consider $|\psi\rangle = |p\rangle$

we expect $\Pi^{+} p \Pi = -p \Rightarrow p\Pi = -\Pi p$

by same argument as for $|x\rangle$ $\quad \Pi |p\rangle = \xi\, |-p\rangle$

How do wavefunctions transform under Π?

$$\psi(\vec{r}) = \langle \vec{r} | \psi \rangle$$

$$\Pi|\psi\rangle \longrightarrow \langle \vec{r}|\, \Pi\, |\psi\rangle = \langle -\vec{r} | \psi \rangle = \psi(-\vec{r})$$

If $|\psi\rangle$ is an eigen~~st~~state of Π, then

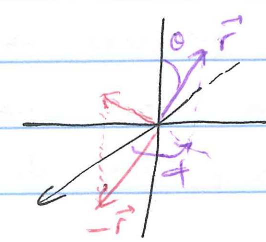
even parity

$$\Pi|\psi\rangle = \pm|\psi\rangle \quad \text{then} \quad \psi(-\vec{r}) = \pm\, \psi(\vec{r})$$

odd parity

Spherical Harmonics under Π

$$\left\{\begin{array}{l} x \to -x \\ y \to -y \\ z \to -z \end{array}\right. \iff \left\{\begin{array}{l} r = \sqrt{x^2+y^2+z^2} \to r \\ \theta \to \pi - \theta \\ \phi \to \phi + \pi \end{array}\right.$$



~~Since~~

$$\cancel{Y_\ell^m = (-1)^m \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, P_\ell^m(\cos\theta)\, e^{im\phi} \quad (\text{for } m \geq 0)}$$

<u>note:</u> $\cos\theta \xrightarrow{P} \cos(\pi-\theta) = -\cos\theta$

$$e^{im\phi} \xrightarrow{P} e^{im(\phi+\pi)} = e^{im\pi} e^{im\phi} = (-1)^m e^{im\phi}$$

One can show that

$$Y_\ell^m(\theta,\phi) \xrightarrow{P} Y_\ell^m(\pi-\theta, \phi+\pi) = (-1)^\ell Y_\ell^m(\theta,\phi)$$

thus $|Y_\ell^m\rangle$ are eigenstates of Π.

Hamiltonian under Π

Most Hamiltonians are <u>even</u> under a parity transformation

for example $H = \frac{P^2}{2m} - \frac{e^2}{R}$ is even

In fact $[H, \Pi] = 0$

Thus we can construct an eigenbasis of H in which the eigenstates are odd and even functions.

(n, ℓ, m) have a definite parity given by ℓ. (⚠ $\alpha|nS\rangle + \beta|nP\rangle$ is an eigenstate of H but does not have a definite parity)

Parity violation:

All known ~~in~~ fundamental interactions conserve parity, except the weak interaction.

gravity, E&M, strong force

All the Hamiltonians that we have studied so far are ~~p~~ even under parity (i.e. Hydrogen).

$$H_H = \frac{P^2}{2m} + \frac{e^2}{R} + \frac{P^4}{8 m_e^3 c^2} + \frac{1}{2}\frac{e^2}{m_e^2 c^2}\frac{1}{R^3}\vec{L}\cdot\vec{S}$$

even   even   even   even   even   even

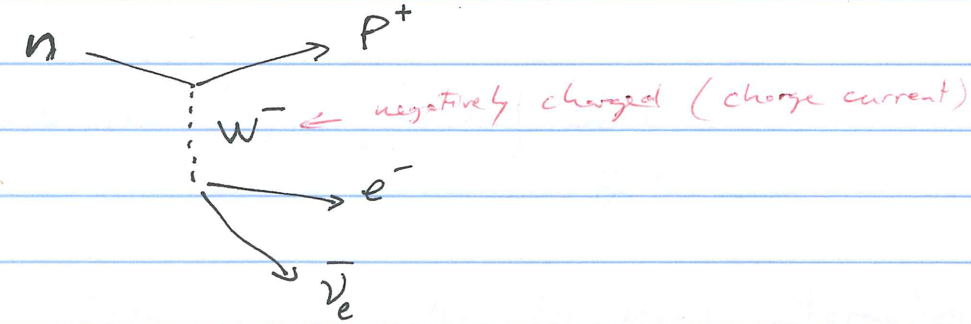
$$+ \frac{A}{\hbar^2}\vec{I}\cdot\vec{J} - \frac{q_e}{2m_e}(\vec{L}+2\vec{S})\cdot\vec{B} - q_e \vec{E}\cdot\vec{R} + \dots$$

even   even   even   even   even   odd   odd   even

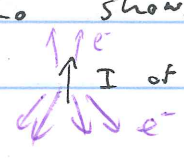
Stark shift appears to violate parity, but only if you don't transform E

The Weak interaction is responsible for radioactive decay

e.g.

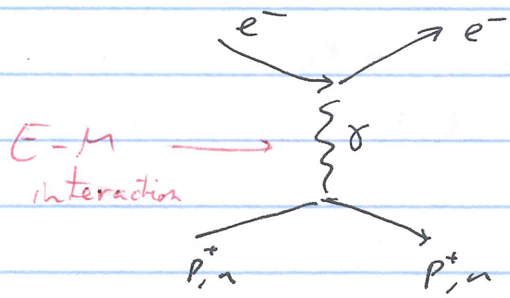


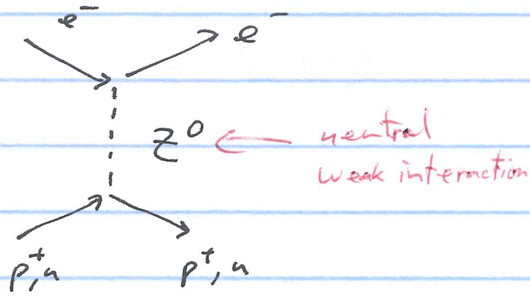
β- decay in ^{60}Co shows an ~~asymmetry~~ asymmetry in the emission of e^-



(Yang, Lee, Wu 1956-57)

There is also a weak neutral current interaction mediated by Z^0





In the non-relativistic limit, this interaction can be written as

$$H_{\cancel{Z}Z^0} = \frac{G_F}{\sqrt{2}} \frac{1}{2m_e c\hbar} Q_w \, \underbrace{\underset{\text{even}}{\overset{e^-\text{ spin}}{\vec{S}}} \cdot \left[\underset{\text{odd}}{\overset{e^-\text{ momentum}}{\vec{P}}} \underset{\text{even}}{\delta^3(\vec{R})} + \underset{\text{even}}{\overset{e^-\text{ position}}{\delta^3(\vec{R})}} \underset{\text{odd}}{\vec{P}} \right]}_{\text{odd}}$$

$G_F \triangleq 3 \times 10^{-12} \, m_e c^2 \left(\frac{\hbar}{m_e c}\right)^3$ and $Q_w = -N + (1 - 4\sin^2\theta_w) Z$

$\simeq -N$

= weak charge of nucleus

~~P~~ $\vec{S} \cdot \vec{P}$ is odd under parity → pseudoscalar

In the standard model $\sin^2\theta_w \simeq 0.23$

(θ_w is called the Weinberg mixing angle)

<u>H_{Z_0} is parity odd</u> but very very small.

~~In principle~~ Thus $[H_0 + H_{Z_0}, \pi] \neq 0$

$\simeq 0$

$\delta^3(\vec{R})$ term indicates that the weak interaction is a contact interaction and only affects the e^- inside the nucleus.