Tuesday, March 26, 2013

Parity violating Hamiltonian in non-relativistic limit: $H_{Z^0} = cst\ \vec{S} \cdot \left[\vec{P}\,\delta^3(\vec{R}) + \delta^3(\vec{R})\,\vec{P}\right]$

H_{Z^0} can be treated using perturbation theory

$$|nS\rangle \longrightarrow |nS\rangle + i\varepsilon|nP\rangle + \ldots = |nS'\rangle$$
$$|nP\rangle \longrightarrow |nP\rangle + i\varepsilon|nS\rangle + \ldots = |nP'\rangle$$

$$\varepsilon \sim 10^{-11} - 10^{-12}$$

Parity forbidden ~~transitions~~ EM transitions now become slightly allowed.

$$H_{EM} = -q_e\,\vec{R}\cdot\vec{E}\cos(\omega t)$$

↳ $\langle nS|Z|(n+1)S\rangle = 0$

but $\langle nS'|Z|(n+1)S'\rangle \neq 0 \simeq 0$.

↳ gives best low energy test of electroweak sector of standard model.

$\vec{S}\cdot\vec{P}$ is a pseudoscalar : – transforms like a scalar under rotations

– Parity odd (scalars are parity even)

$\vec{S}\cdot\vec{P}\,\delta^3(R)$ is also a pseudoscalar

$\vec{S}$ is a pseudovector (axial vector) : – transforms like a geometric vector under rotations (eg. position or velocity vectors)

– Parity even (vectors are parity odd)

[strong relation, total angular momentum selection rules, see Sakurai 3.11.31[3.10.31], 3.11.38[3.10.38] 3.11.40[3.10.40]]

Time Reversal Symmetry

Time reversal operation (or reversal of motion operation)

Classical: time reversal operation $t \to -t$ (see Jackson 6.11 2nd ed.)

examples:

position vector: $\vec{r} \xrightarrow{T} \vec{r}$ (time reversal even)

velocity: $\vec{v} = \frac{d\vec{r}}{dt} \xrightarrow{T} \frac{d\vec{r}}{-dt} = -\vec{v}$ (odd)

momentum: $\vec{p} = m\vec{v} \xrightarrow{T} m(-\vec{v}) = -\vec{p}$ (odd)

angular momentum: $\vec{l} = \vec{r} \times \vec{p} \xrightarrow{T} \vec{r} \times (-\vec{p}) = -\vec{l}$ (odd)

acceleration: $\vec{a} = \frac{d\vec{v}}{dt} \xrightarrow{T} \frac{d(-\vec{v})}{-dt} = \vec{a}$ (even)

Force: $\vec{F} = m\vec{a} \xrightarrow{T} m\vec{a} = \vec{F}$ (even)

Energy: $E = \frac{1}{2}mv^2 \xrightarrow{T} \frac{1}{2}m(-v)^2 = E$ (even)

Electric & magnetic fields: consider Lorentz force law

$$\underbrace{\vec{F}}_{\text{even}} = q\underbrace{\vec{E}}_{\text{even}} + q\underbrace{\vec{v}}_{\text{odd}} \times \underbrace{\vec{B}}_{\text{odd}}$$

$$\Rightarrow \begin{cases} \vec{E} \xrightarrow{T} \vec{E} \text{ (even)} \\ \vec{B} \xrightarrow{T} -\vec{B} \text{ (odd) (i.e. source current: } I \xrightarrow{T} -I) \end{cases}$$

$q \xrightarrow{T,P} q$ (even/even)

$m \xrightarrow{T,P} m$ (even/even)

Quantum Mechanics:

~~Consider~~ we expect that standard QM should respect time reversal symmetry.

(i.e. be odd or even)

Consider the schrodinger equation:

$$i\hbar \frac{\partial}{\partial t} \psi(\vec{r},t) = H \psi(\vec{r},t)$$

Classically, if $\vec{r}(t)$ is a solution of

$$m \frac{d^2\vec{r}}{dt^2} = -\nabla V(\vec{r})$$

then $\vec{r}(-t)$ is also a solution

e.g. $x = \frac{1}{2} g t^2$

if $\psi(\vec{r},t)$ is an eigenfunction: $\psi(\vec{r},t) = \psi_0(\vec{r}) e^{-i\frac{E_0}{\hbar}t}$

then $$i\hbar \frac{\partial}{\partial t} \psi_0(\vec{r}) e^{-i\frac{E_0}{\hbar}t} = \underbrace{H\psi_0(\vec{r})}_{E_0\psi_0(\vec{r})} e^{-i\frac{E_0}{\hbar}t}$$

$-i\frac{E_0}{\hbar} e^{-i\frac{E_0}{\hbar}t}$

$$\cancel{i\hbar\left(\frac{-i}{\hbar}\right)} E_0 \psi_0(\vec{r}) e^{-i\frac{E_0}{\hbar}t} = E_0 \psi_0(\vec{r}) e^{-i\frac{E_0}{\hbar}t}$$

Apply time reversal: $\psi(\vec{r},t) \longrightarrow \psi(\vec{r},-t) = \psi_0(\vec{r}) e^{+i\frac{E_0}{\hbar}t}$

Does it satisfy Schrodinger Eq.?

$$i\hbar \frac{\partial}{\partial t} \psi_0(\vec{r}) e^{+i\frac{E_0}{\hbar}t} \stackrel{?}{=} \underbrace{H\psi_0(\vec{r})}_{E_0\psi_0(\vec{r})} e^{+i\frac{E_0}{\hbar}t}$$

$i\frac{E_0}{\hbar} e^{+i\frac{E_0}{\hbar}t}$

$$i\hbar\left(\frac{i}{\hbar}\right) E_0 \psi_0(\vec{r}) e^{+i\frac{E_0}{\hbar}t} \stackrel{?}{\neq} E_0 \psi_0(\vec{r}) e^{+i\frac{E_0}{\hbar}t}$$

-1

2 resolutions to problem:

1) $t \to -t$ in Schrodinger eqn. then $\psi(\vec{r},-t)$ satisfies

$$-i\hbar \frac{\partial}{\partial t} \psi = H\psi$$

2) Alternate time reversal operation: $\psi(\vec{r},t) \xrightarrow{T} \psi^*(\vec{r},-t) = \psi_0^*(\vec{r}) e^{-i\frac{E_0 t}{\hbar}}$

in eigenbasis $H_0^* = H_0$ since eigenenergies are real $\quad (H\psi_0 = E_0\psi_0)^* \Rightarrow H_0\psi^* = E\psi_0^*$

Time-Reversal operator

we expect the time reversal operator to act according to

$$T\,|\vec{p}\rangle = \underbrace{e^{i\phi}}_{\text{possible phase factor}}\,|-\vec{p}\rangle \quad \leftarrow \text{momentum (i.e. motion) reversed}$$

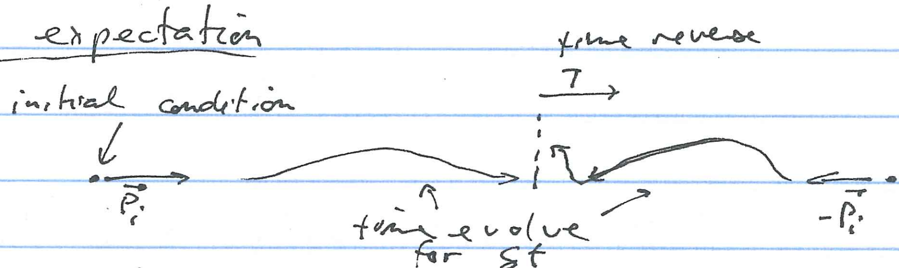
angular momentum: $\underbrace{T\vec{L}T^{-1}}_{\text{time reverse angular momentum}} = -\vec{L} \quad \leftarrow$ angular momentum is reversed (system spins backwards)

Energy: $THT^{-1} = H \iff TH = HT \iff [T, H] = 0$

position: $TRT^{-1} = R$

Kinematic expectation

classical:



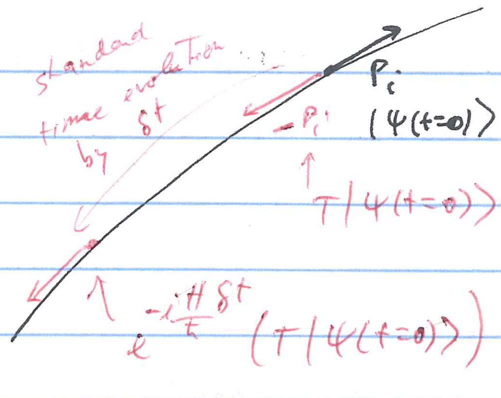
quantum: initial state

$$|\psi(t=0)\rangle \xrightarrow{\text{time evolve}} e^{-i\frac{H\delta t}{\hbar}}\,|\psi(t=0)\rangle$$

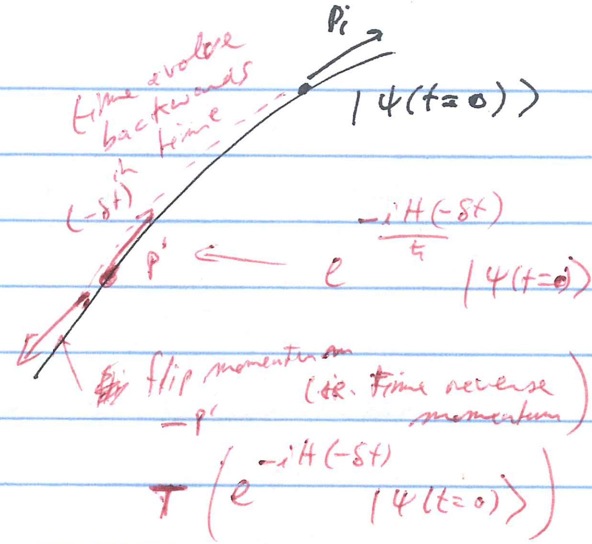
↳ time reverse:

$$T|\psi(t=0)\rangle \xrightarrow{\text{time evolve}} e^{-i\frac{H\delta t}{\hbar}}\left(T|\psi(t=0)\rangle\right)$$

this should be the same as $T\left[e^{-i\frac{H(-\delta t)}{\hbar}}|\psi(t=0)\rangle\right] \equiv e^{-i\frac{H\delta t}{\hbar}}\left(T|\psi(t=0)\rangle\right)$



version #1



version #2

for $\delta t \to 0$, we have

$$T\left[\cancel{1} - i\frac{H(-\delta t)}{\hbar}\right]|\psi(t=0)\rangle \equiv \left[\cancel{1} - i\frac{H\,\delta t}{\hbar}\right]T|\psi(t=0)\rangle$$

$$\Leftrightarrow \quad T\,iH|\psi(t=0)\rangle \equiv -iHT|\psi(t=0)\rangle$$

if this is true for any $|\psi(t=0)\rangle$, then

$$T\,iH \equiv -iHT$$

⚠ If we cancel "i"'s, then $TH = -HT$

$$\Leftrightarrow \quad TH + HT = 0$$

$$\Rightarrow \quad [T, H] \neq 0$$

contradicts expectations of $[H,T]=0$

$\Rightarrow$ we cannot cancel the "i"'s $\Rightarrow$ T is not a linear operator

In fact, T is <u>anti-linear</u> !!!

... and also <u>anti-unitary</u>!

definition: Anti-linear operator

An operator A is said to be anti-linear if

$$A\left(c_1|\alpha\rangle + c_2|\beta\rangle\right) = c_1^* A|\alpha\rangle + c_2^* A|\beta\rangle$$

stopped here

note: $T\left(c_1|\alpha\rangle + c_2|\beta\rangle\right) = c_1^* \underbrace{T|\alpha\rangle} + c_2^* \underbrace{T|\beta\rangle}$ to be determined

definition: Anti-unitary operator

An anti-linear operator A is said to be anti-linear if

$$\langle\tilde{\beta}|\tilde{\alpha}\rangle = \langle\beta|\alpha\rangle^*$$

unitary operator has $\langle\tilde{\beta}|\tilde{\alpha}\rangle = \langle\beta|\alpha\rangle$

where
$$A|\alpha\rangle = |\tilde{\alpha}\rangle$$
$$A|\beta\rangle = |\tilde{\beta}\rangle$$

note: Norm and probability are still preserved

$$p = |\langle\tilde{\beta}|\tilde{\alpha}\rangle|^2 = |\langle\beta|\alpha\rangle|^2$$

⚠ $$\langle\beta|T|\alpha\rangle = \langle\beta|\left(T|\alpha\rangle\right)$$

do not try to do $\cancel{(\langle\beta|T)|\alpha\rangle}$

Position Space wave-functions

Consider the position space wave function $\psi(\vec{r}, t=0) = \langle\vec{r}|\psi\rangle$

$$|\psi\rangle = \int d^3r \langle\vec{r}|\psi\rangle |\vec{r}\rangle$$

then $$T|\psi\rangle = \int d^3r \langle\vec{r}|\psi\rangle^* \underbrace{T|\vec{r}\rangle}_{|\vec{r}\rangle}$$

$T|\vec{r}\rangle = |\vec{r}\rangle$ ← up to some phase factor which we set to 1

$$\Rightarrow T|\psi\rangle = \int d^3r \langle\vec{r}|\psi\rangle^* |\vec{r}\rangle$$

thus $$\boxed{\underset{(t=0)}{\psi(\vec{r})} \xrightarrow{T} \underset{(t=0)}{\psi^*(\vec{r})}}$$