~~Thurs~~ Thursday, March 28, 2013

#1

Time Reversal Symmetry (or Reversal of motion symmetry) (continued)

definition : Anti-linear operator

An operator A is said to be anti-linear if

$$A\left(c_1|\alpha\rangle + c_2|\beta\rangle\right) = c_1^* A|\alpha\rangle + c_2^* A|\beta\rangle$$

note: $T\left(c_1|\alpha\rangle + c_2|\beta\rangle\right) = c_1^* \underbrace{T|\alpha\rangle} + c_2^* \underbrace{T|\beta\rangle}$ to be determined

definition : Anti-unitary operator

An anti-linear operator A is said to be anti-linear if

$$\langle\tilde{\beta}|\tilde{\alpha}\rangle = \langle\beta|\alpha\rangle^*$$

unitary operator has $\langle\tilde{\beta}|\tilde{\alpha}\rangle = \langle\beta|\alpha\rangle$

where $A|\alpha\rangle = |\tilde{\alpha}\rangle$

$A|\beta\rangle = |\tilde{\beta}\rangle$

note: Norm and probability are still preserved

$$p = |\langle\tilde{\beta}|\tilde{\alpha}\rangle|^2 = |\langle\beta|\alpha\rangle|^2$$

⚠ $\langle\beta|T|\alpha\rangle = \langle\beta|\left(T|\alpha\rangle\right)$

do not try to do $\cancel{\left(\langle\beta|T\right)|\alpha\rangle}$

Position space wave-functions

Consider the position space wavefunction $\Rightarrow \psi(\vec{r}, t=0) = \langle\vec{r}|\psi\rangle$

$$|\psi\rangle = \int d^3r \langle\vec{r}|\psi\rangle|\vec{r}\rangle$$

T = "reversal of motion" operator

then $T|\psi\rangle = \int d^3r \langle\vec{r}|\psi\rangle^* \underbrace{T|\vec{r}\rangle}_{|\vec{r}\rangle}$ $\quad T|\vec{r}\rangle = |\vec{r}\rangle$

← up to some phase factor which we set to 1

$\Rightarrow T|\psi\rangle = \int d^3r \langle\vec{r}|\psi\rangle^* |\vec{r}\rangle$

thus $\boxed{\underset{(t=0)}{\psi(\vec{r})} \xrightarrow{T} \underset{(t=0)}{\psi^*(\vec{r})}}$

~~Position Momentum space wavefunctions~~

note: ~~...~~ If $|\psi_n\rangle$ is a non-degenerate eigenstate of H,

then $H\left(T|\psi_n\rangle\right) = T H |\psi_n\rangle = T E_n |\psi_n\rangle = E_n\left(T|\psi_n\rangle\right)$

$\Rightarrow$ $T|\psi_n\rangle$ is also an eigenstate of H with eigenenergy E_n, but $\{|\psi_n\rangle, E_n\}$ is non-degenerate $\Rightarrow T|\psi_n\rangle = |\psi_n\rangle$

$\Rightarrow \psi_n(\vec{r}) = \psi_n^*(\vec{r}) \Rightarrow \psi_n(\vec{r})$ is real

(for spinless particles)

Momentum Space wavefunctions

Consider the position space wavefunction $\psi(\vec{p}, t=0) = \langle \vec{p} | \psi \rangle$

$$|\psi\rangle = \int d^3p \, \langle \vec{p} | \psi \rangle \, |\vec{p}\rangle$$

"reversal of motion" operator

$$T|\vec{p}\rangle = |-\vec{p}\rangle$$

then
$$T|\psi\rangle = \int d^3p \, \langle \vec{p} | \psi \rangle^* \underbrace{T|\vec{p}\rangle}_{|-\vec{p}\rangle}$$

$$= \int d^3p \, \langle p | \psi \rangle^* \, |-\vec{p}\rangle$$

substitution $\vec{k} = -\vec{p}$, $d^3k = -d^3p$

$$= \int d^3p \, \langle -p | \psi \rangle^* \, |\vec{p}\rangle$$

$$\Rightarrow \boxed{\underset{(t=0)}{\psi(\vec{p})} \xrightarrow{T} \underset{t=0}{\psi^*(-\vec{p})}}$$

⚠ Action of T (time reversal operator) depends on basis / representation.

Angular momentum wavefunction

case 1: ~~...~~ orbital angular momentum wavefunctions

$(e^{im\phi})^* = e^{i(-m)\phi}$

$$\psi(\vec{r}) = Y_\ell^m(\theta,\phi) \xrightarrow{T} Y_\ell^{m\,*}(\theta,\phi) = (-1)^m \, Y_\ell^{-m}(\theta,\phi)$$

real

due to $(-1)^m$ factor already present in Y_ℓ^m

thus $T|l,m\rangle = (-1)^m |l,-m\rangle$

note: $T^2|l,m\rangle = |l,m\rangle$, also $\psi(\vec{r}) \xrightarrow{T^2} \psi(\vec{r})$.

Spin-1/2 wavefunctions

$T|\uparrow\rangle = ?$ or $T|\downarrow\rangle = ?$

We will admit that anti-unitary operators can be written in the form $A = UK$, where U is unitary (linear) and K perform the complex conjugation operation:

$$K\left(c_1|\alpha\rangle + c_2|\beta\rangle\right) = c_1^* K|\alpha\rangle + c_2^* K|\beta\rangle$$
$$= c_1^*|\alpha\rangle + c_2^*|\beta\rangle$$

(if $|\alpha\rangle$ and $|\beta\rangle$ are basis kets)

So $T = U_T K$

complex conjugator acts to the right (on a ket)

Unitary operator for time reversal (or reversal of motion)

we know $T\vec{S}T^{-1} = -\vec{S}$

$$\Rightarrow T S_i T^{-1} = -S_i$$
$$\Leftrightarrow U_T K S_i (U_T K)^{-1} = -S_i$$
$$\Leftrightarrow U_T K S_i K^{-1} U_T^{-1} = -S_i \quad (\text{clearly } K^{-1} = K)$$

we note that in S_z eigenbasis we can write $\{|\uparrow\rangle, |\downarrow\rangle\}$

$$S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$K S_z K = S_z K^2 = S_z$ | $K S_x K = S_x K^2 = S_x$ | $K S_y K = -S_y K^2 = -S_y$

$\Rightarrow T S_z T^{-1} = -S_z$

$U_T K S_z K U_T^{-1} = -S_z \Rightarrow U_T S_z U_T^{-1} = -S_z$ | $U_T S_x U_T^{-1} = -S_x$ | $U_T S_y U_T^{-1} = S_y$

some phase ϕ

One can show that $U_T = e^{i\phi}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ satisfies these ~~requirements~~ requirements

set $\phi = 0$ $\quad = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$

$T|\uparrow\rangle = |\downarrow\rangle$

$T|\downarrow\rangle = -|\uparrow\rangle$

If we apply T twice: $TT = T^2 =$

$$T^2 = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} K \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} K = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} K^2$$

$$= \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} = \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix} = -\mathbb{1}$$

thus $T^2|\uparrow\rangle = -|\uparrow\rangle$

$T^2|\downarrow\rangle = -|\downarrow\rangle$

If you time reverse a spin-$\frac{1}{2}$ twice, you don't get the same thing back

but, $T^4 = 1$

Surprising? sort of, but recall that the rotation operator is

$$\mathcal{D}(\theta\phi) = \exp\left(-i\frac{\vec{J}\cdot\hat{n}\,\theta}{\hbar}\right)$$

for $\hat{n} = \hat{z}$ in a 2-D system ("spin-$\frac{1}{2}$")

$$\mathcal{D}(\theta) = \exp\left(-i\frac{S_z\theta}{\hbar}\right) = \begin{array}{c|c} & |\uparrow\rangle \quad\quad |\downarrow\rangle \\ \begin{matrix}\langle\uparrow| \\ \langle\downarrow|\end{matrix} & \begin{pmatrix}e^{-i\theta/2} & 0\\ 0 & e^{+i\theta/2}\end{pmatrix}\end{array}$$

$\mathcal{D}(\theta = 2\pi)|\uparrow\rangle = -|\uparrow\rangle$

but $\mathcal{D}(\theta = 4\pi)|\uparrow\rangle = +|\uparrow\rangle$

More generally, one can show that

$$T^2 \,|\, j\text{-half-integer}\rangle = -\,|\, j\text{-half-integer}\rangle$$
$$T^2 \,|\, j\text{-integer}\rangle = +\,|\, j\text{-integer}\rangle$$

If we change our phase convention to $\phi = \pi/2$

$$U_T = e^{i\pi/2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

then

$$T\,|j,m\rangle = i^{2m}\,|j,-m\rangle$$

& (which is consistent with $T\,|Y_\ell^m\rangle = (-1)^m\,|Y_\ell^{-m}\rangle$)

this phase convention is not consistent with the one that gives $J_\pm |j,m\rangle = \hbar\sqrt{(j\mp m)(j\pm m+1)}\,|j,m\pm 1\rangle$

T- violation

~~[illegible]~~ Generally the Hamiltonian is T-even (i.e. $THT^{-1} = H$ or $[T,H] = 0$), but the ~~[illegible]~~ Standard Model and extensions to it (e.g. ~~[illegible]~~ supersymmetry) predict T-odd Hamiltonian terms, including an electron dipole moment $\vec{d}_e \propto \vec{S}$ so that

$$H_{edm} = -\vec{d}_e \cdot \vec{E} = cst\ \vec{S}\cdot\vec{E}$$

$\vec{E}_{total}$ (not $\vec{E}_{external}$)

[see Sakurai problem# 5.15]

note: H_{edm} ~~[illegible]~~ is also parity odd $\pi H_{edm} + H_{edm}\pi = 0$

method: Look for an $\vec{E}$ − ~~[illegible]~~ dependent Zeeman shift → so far no EDM detected